Sub-riemannian geometry from intrinsic viewpoint

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Abstract

Gromov proposed to extract the (differential) geometric content of a sub-riemannian space exclusively from its Carnot-Carathéodory distance. One of the most striking features of a regular sub-riemannian space is that it has at any point a metric tangent space with the algebraic structure of a Carnot group, hence a homogeneous Lie group. Siebert characterizes homogeneous Lie groups as locally compact groups admitting a contracting and continuous one-parameter group of automorphisms. Siebert result has not a metric character.

In these notes I show that sub-riemannian geometry may be described by about 12 axioms, without using any a priori given differential structure, but using dilation structures instead. Dilation structures bring forth the other intrinsic ingredient, namely the dilations, thus blending Gromov metric point of view with Siebert algebraic one.

MSC2000: 51K10, 53C17, 53C23

1 Foreword

In these notes I show that sub-riemannian geometry may be described by about 12 axioms, **without using any a priori given differential structure**. In my opinion this shows the power of the dilation structures approach.

A geometry is not a simple object, for example euclidean geometry needs twice this number of axioms. It should be clear that renouncing to such a basic object as a differential structure is payed by the introduction of a number of axioms which might seem too high at the first view. It is not so high though; just for an example, the number of axioms for the euclidean geometry decreases dramatically once we use as basic objects the algebraic and topological structure of real numbers (or real vector spaces).

From Gromov viewpoint, the only intrinsic object of a sub-riemannian space is the Carnot-Carathéodory distance. One of the most striking features of a regular sub-riemannian space is that it has at any point a metric tangent space, which algebraically is a Carnot group. This has been proved several times, by using the CC distance and lots of informations coming from the underlying differential structure of the manifold. Let us compare this with the result of Siebert, which characterizes homogeneous Lie groups as locally compact groups admitting a contracting and continuous one-parameter group of automorphisms. Siebert result has not a metric character.

In the work presented in this paper we tried to argue that we need more than only the CC distance in order to describe regular sub-riemannian manifolds, but less than the underlying differential structure: we need only dilation structures. Dilation structures bring forth the other intrinsic ingredient, namely the dilations, which are generalizations of Siebert' contracting group of automorphisms.

As it is maybe to be expected from a course notes paper, these notes are build from previous papers of mine, the only new section is the one on curvdimension. I hope that the unitary presentation will help the understanding of the subject.

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2 Metric spaces, groupoids, norms

Definition 2.1 *A metric space (X, d) is a set X endowed with a distance function $d : X \times X \to [0, +\infty)$. In the metric space (X, d), the distance between two points $x, y \in X$ is $d(x, y) \geq 0$. The distance d satisfies the following axioms:*

(i) $d(x, y) = 0$ *if and only if* $x = y$,

(ii) (symmetry) for any $x, y \in X$ $d(x, y) = d(y, x)$,

(iii) (triangle inequality) for any $x, y, z \in X$ $d(x, z) \leq d(x, y) + d(y, z)$.

The ball of radius $r > 0$ *and center* $x \in X$ *is the set*

$$B(x, r) = \{y \in X : d(x, y) < r\} \quad .$$

Sometimes we shall use the notation $B_d(x, r)$ for the ball of center x and radius r with respect to the distance d, in order to emphasize the dependence on the distance d. Any metric space (X, d) is endowed with the topology generated by balls. The notations $\bar{B}(x, r)$ and $\bar{B}_d(x, r)$ are used for the closed ball centered at x, with radius r.

A pointed metric space (X, x, d) is a metric space (X, d) with a chosen point $x \in X$.

Metric spaces have been introduced by Fréchet in the paper [Sur quelques points du calcul fonctionnel, *Rendic. Circ. Mat. Palermo 22* (1906), 1-74].

2.1 Normed groups and normed groupoids

Any normed vector space is a metric space. More generally, an abelian group with a norm is a metric space; indeed, by looking at the normed vector space example we see that in order to define the distance we need the norm function and the abelian group structure of the vector space. (Later in this paper, he multiplication by scalars will provide us with the first example of a metric space with

dilations). This leads us to the introduction of normed groups. Let us give, in increasing generality, the definition of a normed group, then the definition of a normed groupoid.

Definition 2.2 *A normed group (G, ρ) is a pair formed by:*

- *a group G, with the operation $(x, y) \in G \times G \mapsto xy$, inverse denoted by $x \in G \mapsto x^{-1}$ and neutral element denoted by e,*
- *a norm function $\rho : G \to [0, +\infty)$, which satisfies the following axioms:*

  (i) $\rho(x) = 0$ if and only if $x = e$,

  (ii) (symmetry) for any $x \in G$ $\rho(x^{-1}) = \rho(x)$,

  (iii) (sub-additivity) for any $x, y \in G$ $\rho(xy) \leq \rho(x) + \rho(y)$.

Any normed group (G, ρ) can be seen as a metric space, with any of the distances

$$d_L(x, y) = \rho(x^{-1}y) \quad , \quad d_R(x, y) = \rho(xy^{-1}) \quad .$$

The function d_L is *left*-invariant, i.e. for any $x, y, z \in G$ we have $d_L(zx, zy) = d_L(x, y)$. Similarly d_R is *right*-invariant, that is for any $x, y, z \in G$ we have $d_R(xz, yz) = d_R(x, y)$.

Indeed, the first axiom of a distance is a consequence of the first axiom of a norm, the symmetry axiom for distances is a consequence of the symmetry axiom of the norm and the triangle inequality comes from the group identity

$$x^{-1}z = \left(x^{-1}y\right)\left(y^{-1}z\right)$$

(which itself is a consequence of the associativity of the group operation and of the existence of inverse) and from the sub-additivity of the norm. The left-invariance of d_L comes from the group identity $(zx)^{-1}(zy) = x^{-1}y$.

Groupoids are generalization of groups. A groupoid is a small category such that any arrow is invertible. The set of arrows of a groupoid is a set with a partially defined binary operation and a unary operation (the inverse function), which satisfy several properties. A norm is then a function defined on the set of arrows of a groupoid, with properties similar with the ones of a norm over a group. This is the definition which we give further.

Definition 2.3 *A normed groupoid (G, ρ) is a pair formed by:*

- *a groupoid G, which is a set with two operations $inv : G \to G$, $m : G^{(2)} \subset G \times G \to G$, which satisfy a number of properties. With the notations $inv(a) = a^{-1}$, $m(a, b) = ab$, these properties are: for any $a, b, c \in G$*

  (i) if $(a, b) \in G^{(2)}$ and $(b, c) \in G^{(2)}$ then $(a, bc) \in G^{(2)}$ and $(ab, c) \in G^{(2)}$ and we have $a(bc) = (ab)c$,

  (ii) $(a, a^{-1}) \in G^{(2)}$ and $(a^{-1}, a) \in G^{(2)}$,

  (iii) if $(a, b) \in G^{(2)}$ then $abb^{-1} = a$ and $a^{-1}ab = b$.

  The set $X = Ob(G)$ is formed by all products $a^{-1}a$, $a \in G$. For any $a \in G$ we let $\alpha(a) = a^{-1}a$ and $\omega(a) = aa^{-1}$.

- *a norm function $d : G \to [0, +\infty)$ which satisfies the following axioms:*

  (i) $d(g) = 0$ if and only if $g \in Ob(G)$,

(ii) (symmetry) for any $g \in G$, $d(g^{-1}) = d(g)$,

(iii) (sub-additivity) for any $(g,h) \in G^{(2)}$, $d(gh) \leq d(g) + d(h)$,

If $Ob(G)$ is a singleton then G is just a group and the previous definition corresponds exactly to the definition 2.2 of a normed group. As in the case of normed groups, normed groupoids induce metric spaces too.

Proposition 2.4 *Let* (G,d) *be a normed groupoid and* $x \in Ob(G)$. *Then the space* $(\alpha^{-1}(x), d_x)$ *is a metric space, with the distance* d_x *defined by: for any* $g,h \in G$ *with* $\alpha(g) = \alpha(h) = x$ *we have* $d_x(g,h) = d(gh^{-1})$.

Therefore a normed groupoid can be seen as a disjoint union of metric spaces

$$G = \bigcup_{x \in Ob(G)} \alpha^{-1}(x) \quad , \tag{1}$$

with the property that right translations in the groupoid are isometries, that is: for any $u \in G$ *the transformation*

$$R_u : \alpha^{-1}(\omega(u)) \to \alpha^{-1}(\alpha(u)) \quad , \quad R_u(g) = gu$$

has the property for any $g,h \in \alpha^{-1}(\omega(u))$

$$d_{\omega(u)}(g,h) = d_{\alpha(u)}(R_u(g), R_u(h)) \quad .$$

Proof. We begin by noticing that if $\alpha(g) = \alpha(h)$ then $(g,h^{-1}) \in G^{(2)}$, therefore the expression gh^{-1} makes sense. The rest of the proof of the first part of the proposition is identical with the proof of the previous proposition.

For the proof of the second part of the proposition remark first that R_u is well defined and that

$$R_u(g)\left(R_u(h)\right)^{-1} = gh^{-1} \quad .$$

Then we have:

$$d_{\alpha(u)}(R_u(g), R_u(h)) = d\left(R_u(g)\left(R_u(h)\right)^{-1}\right) =$$

$$= d(gh^{-1}) = d_{\omega(u)}(g,h) \quad .$$

□

Therefore normed groupoids provide examples of (disjoint unions of) metric spaces. Are there metric spaces more general than these? No, in fact we have the following.

Proposition 2.5 *Any metric space can be constructed from a normed groupoid, as in proposition 2.4. Precisely, let* (X,d) *be a metric space and consider the* trivial groupoid $G = X \times X$ *with multiplication*

$$(x,y)(y,z) = (x,z)$$

and inverse $(x,y)^{-1} = (y,x)$. *Then* (G,d) *is a normed groupoid and moreover any component of the decomposition (1) of* G *is isometric with* (X,d).

Conversely, if $G = X \times X$ *is the trivial groupoid associated to the set* X *and* d *is a norm on* G *then* (X,d) *is a metric space.*

Proof. We begin by noticing that $\alpha(x,y)=(y,y)$, $\omega(x,y)=(x,x)$, therefore $Ob(G)=\{(x,x):x\in X\}$ can be identified with X by the bijection $(x,x)\mapsto x$. Moreover, for any $x\in X$ we have

$$\alpha^{-1}((x,x))=X\times\{x\} \quad .$$

Because $d:X\times X\to[0,+\infty)$ and $G=X\times X$ it follows that $d:G\to[0,+\infty)$. We have to check the properties of a norm over a groupoid. But these are straightforward. The statement (i) ($d(x,y)=0$ if and only if $(x,y)\in Ob(G)$) is equivalent with $d(x,y)=0$ if and only if $x=y$. The symmetry condition (ii) is just the symmetry of the distance: $d(x,y)=d(y,x)$. Finally the sub-additivity of d seen as defined on the groupoid G is equivalent with the triangle inequality:

$$d((x,y)(y,z))=d(x,z)\leq d(x,y)+d(y,z) \quad .$$

In conclusion (G,d) is a normed groupoid if and only if (X,d) is a metric space.

For any $x\in X$ the distance $d_{(x,x)}$ on the space $\alpha^{-1}((x,x))$ has the expression:

$$d_{(x,x)}((u,x),(v,x))=d((u,x)(v,x)^{-1})=d((u,x)(x,v))=d(u,v)$$

therefore the metric space $(\alpha^{-1}((x,x)),d_{(x,x)})$ is isometric with (X,d) by the isometry $(u,x)\mapsto u$, for any $u\in X$. $\square$

In conclusion normed groups give particular examples of metric spaces and metric spaces are particular examples of normed groupoids. For this reason normed groups make good examples of metric spaces. It is also interesting to extend the theory of metric spaces to normed groupoids (other than trivial normed groupoids). This was started in [12].

2.2 Gromov-Hausdorff distance. Metric tangent spaces

Let us denote by CMS the set of isometry classes of pointed compact metric spaces. The distance on this set is the Gromov distance between (isometry classes of) pointed metric spaces and the topology is induced by this distance.

We may introduce the Gromov-Haudorff distance by way of (cartographic like) maps.

Definition 2.6 *Let $\rho\subset X\times Y$ be a relation which represents a map of $dom\ \rho\subset(X,d)$ into $im\ \rho\subset(Y,D)$. To this map are associated three quantities: accuracy, precision and resolution.*

The accuracy of the map ρ is defined by:

$$acc(\rho)=\sup\{\mid D(y_1,y_2)-d(x_1,x_2)\mid\ :(x_1,y_1)\in\rho\,,\,(x_2,y_2)\in\rho\} \tag{2}$$

The resolution of ρ at $y\in im\ \rho$ is

$$res(\rho)(y)=\sup\{d(x_1,x_2)\ :(x_1,y)\in\rho\,,\,(x_2,y)\in\rho\} \tag{3}$$

and the resolution of ρ is given by:

$$res(\rho)=\sup\{res(\rho)(y)\ :y\in im\ \rho\} \tag{4}$$

The precision of ρ at $x\in dom\ \rho$ is

$$prec(\rho)(x)=\sup\{D(y_1,y_2)\ :(x,y_1)\in\rho\,,\,(x,y_2)\in\rho\} \tag{5}$$

and the precision of ρ is given by:

$$prec(\rho)=\sup\{prec(\rho)(x)\ :x\in dom\ \rho\} \tag{6}$$

We may need to perform also a "cartographic generalization", starting from a relation ρ, with domain $M = dom(\rho)$ which is ε-dense in (X, d), then we perform a " cartographic generalization". The result is a relation $\bar{\rho}$ as in the following definition.

Definition 2.7 *A subset $M \subset X$ of a metric space (X, d) is ε-dense in X if for any $u \in X$ there is $x \in M$ such that $d(x, u) \leq \varepsilon$.*

Let $\rho \subset X \times Y$ be a relation such that dom ρ is ε-dense in (X, d) and im ρ is μ-dense in (Y, D). We define then $\bar{\rho} \subset X \times Y$ by: $(x, y) \in \bar{\rho}$ if there is $(x', y') \in \rho$ such that $d(x, x') \leq \varepsilon$ and $D(y, y') \leq \mu$.

If ρ is a relation as described in definition 2.7 then accuracy $acc(\rho)$, ε and μ control the precision $prec(\rho)$ and resolution $res(\rho)$. Moreover, the accuracy, precision and resolution of $\bar{\rho}$ are controlled by those of ρ and ε, μ, as well. This is explained in the next proposition.

Proposition 2.8 *Let ρ and $\bar{\rho}$ be as described in definition 2.7. Then:*

(a) $res(\rho) \leq acc(\rho)$,

(b) $prec(\rho) \leq acc(\rho)$,

(c) $res(\rho) + 2\varepsilon \leq res(\bar{\rho}) \leq acc(\rho) + 2(\varepsilon + \mu)$,

(d) $prec(\rho) + 2\mu \leq prec(\bar{\rho}) \leq acc(\rho) + 2(\varepsilon + \mu)$,

(e) $\mid acc(\bar{\rho}) - acc(\rho) \mid \leq 2(\varepsilon + \mu)$.

Proof. Remark that (a), (b) are immediate consequences of definition 2.6 and that (c) and (d) must have identical proofs, just by switching ε with μ and X with Y respectively. I shall prove therefore (c) and (e).

For proving (c), consider $y \in Y$. By definition of $\bar{\rho}$ we write

$$\{x \in X : (x, y) \in \bar{\rho}\} = \bigcup_{(x', y') \in \rho, y' \in \bar{B}(y, \mu)} \bar{B}(x', \varepsilon)$$

Therefore we get

$$res(\bar{\rho})(y) \geq 2\varepsilon + \sup\left\{res(\rho)(y') : y' \in im(\rho) \cap \bar{B}(y, \mu)\right\}$$

By taking the supremum over all $y \in Y$ we obtain the inequality

$$res(\rho) + 2\varepsilon \leq res(\bar{\rho})$$

For the other inequality, let us consider $(x_1, y), (x_2, y) \in \bar{\rho}$ and $(x_1', y_1'), (x_2', y_2') \in \rho$ such that $d(x_1, x_1') \leq \varepsilon, d(x_2, x_2') \leq \varepsilon, D(y_1', y) \leq \mu, D(y_2', y) \leq \mu$. Then:

$$d(x_1, x_2) \leq 2\varepsilon + d(x_1', x_2') \leq 2\varepsilon + acc(\rho) + d(y_1', y_2') \leq 2(\varepsilon + \mu) + acc(\rho)$$

Take now a supremum and arrive to the desired inequality.

For the proof of (e) let us consider for $i = 1, 2$ $(x_i, y_i) \in \bar{\rho}, (x_i', y_i') \in \rho$ such that $d(x_i, x_i') \leq \varepsilon, D(y_i, y_i') \leq \mu$. It is then enough to take absolute values and transform the following equality

$$d(x_1, x_2) - D(y_1, y_2) = d(x_1, x_2) - d(x_1', x_2') + d(x_1', x_2') - D(y_1', y_2') +$$
$$+ D(y_1', y_2') - D(y_1, y_2)$$

into well chosen, but straightforward, inequalities. □

We have all the tools needed to define the Gromov-Hausdorff distance as the optimal lower bound for the accuracy of maps of (X, d) into (Y, D).

Definition 2.9 *Let (X,d), (Y,D), be a pair of metric spaces and $\mu > 0$. We shall say that μ is admissible if there is a relation $\rho \subset X \times Y$ such that dom $\rho = X$, im $\rho = Y$, and $acc(\rho) \leq \mu$. The Gromov-Hausdorff distance between (X,d) and (Y,D) is the infimum of admissible numbers μ.*

This is a true distance (between isometry classes of pointed metric spaces) in the case (X,d) and (Y,D) are compact.

To any locally compact metric space we can associate a metric profile [10, 11]. This metric profile is a way of organizing the information given by the distance function in order to get an understanding of the local behaviour of the distance around a point of the space. We need to consider local compactness in order to compact small balls in the next definition.

Definition 2.10 *The metric profile associated to the locally metric space (M,d) is the assignment (for small enough $\varepsilon > 0$)*

$$(\varepsilon > 0,\ x \in M) \ \mapsto\ \mathbb{P}^m(\varepsilon, x) = \left[\bar{B}(x,1), \frac{1}{\varepsilon}d, x\right] \in CMS$$

We may see the metric profile as a bundle of curves in CMS over the metric space. The metric profile is not any curve in CMS. Indeed, for any $\varepsilon, b > 0$, sufficiently small, we have

$$\mathbb{P}^m(\varepsilon b, x) = \mathbb{P}^m_{d_b}(\varepsilon, x)$$

where $d_b = (1/b)d$ and $\mathbb{P}^m_{d_b}(\varepsilon, x) = [\bar{B}(x,1), \frac{1}{\varepsilon}d_b, x]$.

These curves give interesting local and infinitesimal information about the metric space. For example, what kind of metric space has constant metric profile with respect to one of its points?

Definition 2.11 *A metric cone (X,d,x) is a locally compact metric space (X,d), with a marked point $x \in X$ such that for any $a, b \in (0,1]$ we have*

$$\mathbb{P}^m(a,x) = \mathbb{P}^m(b,x)$$

Metric cones are self-similar, in the sense that they have dilations.

Definition 2.12 *Let (X,d,x) be a metric cone. For any $\varepsilon \in (0,1]$ a dilation is a function $\delta^x_\varepsilon : \bar{B}(x,1) \to \bar{B}(x,\varepsilon)$ such that*

- $\delta^x_\varepsilon(x) = x$,
- *for any $u, v \in X$ we have*

$$d\left(\delta^x_\varepsilon(u), \delta^x_\varepsilon(v)\right) = \varepsilon\, d(u,v)$$

The existence of dilations for metric cones comes from the definition 2.11. Indeed, dilations are just isometries from $(\bar{B}(x,1), d, x)$ to $(\bar{B}, \frac{1}{a}d, x)$.

Metric cones are good candidates for being tangent spaces in the metric sense. This is related to the next natural question: is the metric profile $\mathbb{P}^m(\varepsilon, x)$ converging when ε goes to 0?

Definition 2.13 *A (locally compact) metric space (M,d) admits a (metric) tangent space in $x \in M$ if the associated metric profile $\varepsilon \mapsto \mathbb{P}^m(\varepsilon, x)$ admits a prolongation by continuity in $\varepsilon = 0$, i.e if the following limit exists:*

$$[T_xM, d^x, x] = \lim_{\varepsilon \to 0} \mathbb{P}^m(\varepsilon, x) \tag{7}$$

The connection between metric cones, tangent spaces and metric profiles in the abstract sense is made by the following proposition.

Proposition 2.14 *Metric tangent spaces are metric cones.*

Proof. A tangent space $[V, d_v, v]$ exists if and only if we have the limit from the relation (7). In this case there exists a prolongation by continuity to $\varepsilon = 0$ of the metric profile $\mathbb{P}^m(\cdot, x)$. For any $a \in (0, 1]$ we have

$$\left[\bar{B}(x,1), \frac{1}{a}d^x, x\right] = \lim_{\varepsilon \to 0} \mathbb{P}^m(a\varepsilon, x)$$

Therefore

$$\left[\bar{B}(x,1), \frac{1}{a}d^x, x\right] = [T_xM, d^x, x]$$

□

We shall need further a notion of an abstract metric profile.

Definition 2.15 *An abstract metric profile is a curve $\mathbb{P} : [0, a] \to CMS$ such that*

(a) it is continuous at 0,
(b) for any $b \in [0, a]$ and $\varepsilon \in (0, 1]$ we have

$$d_{GH}(\mathbb{P}(\varepsilon b), \mathbb{P}^m_{d_b}(\varepsilon, x_b)) \ = \ O(\varepsilon)$$

The function $\mathcal{O}(\varepsilon)$ may change with b. We used the notations

$$\mathbb{P}(b) = [\bar{B}(x,1), d_b, x_b] \quad \text{and} \quad \mathbb{P}^m_{d_b}(\varepsilon, x) = \left[\bar{B}(x,1), \frac{1}{\varepsilon}d_b, x_b\right]$$

2.3 Curvdimension and curvature

In the case of a riemannian manifold (X, g), with smooth enough (typically $\mathcal{C}^1$) metric g, the tangent metric spaces exist for any point of the manifold. Ideed, the tangent metric space $[T_xX, d^x, x]$ is the isometry class of a small neighbourhood of the origin of the tangent space (in differential geometric sense) T_xX, with d^x being the euclidean distange induced by the norm given by g_x. Moreover, we have the following description of the sectional curvature.

Proposition 2.16 *Let (X, d) be a $\mathcal{C}^4$ smooth riemannian manifold with d the length distance induced by the riemannian metric g. Suppose that for a point $x \in X$ the sectional curvature is bounded in the sense that for any linearly independent $u, v \in T_xX$ we have $\mid K_x(u, v) \mid \leq C$. Then for any sufficiently small $\varepsilon > 0$ we have*

$$\frac{1}{\varepsilon^2} d_{GH}(\mathbb{P}^m(\varepsilon, x), [T_xX, d^x, x]) \ \leq \ \frac{1}{3}C + \mathcal{O}(\varepsilon) \tag{8}$$

Proof. This is well known, in another form. Indeed, for small enough ε, consider the geodesic exponential map which associates to any $u \in W \subset T_xX$ (in a neighbourhood W of the origin which is independent of ε) the point $\exp_x \varepsilon u$. Define now the distance

$$d^x_\varepsilon(u, v) \ = \ \frac{1}{\varepsilon} d(\exp_x \varepsilon u, \exp_x \varepsilon v)$$

We can choose the neighbourhood W to be the unit ball with respect to the distance d^x in order to get the following estimate:

$$\sup\{\mid d^x_\varepsilon(u, v) - d^x(u, v) \mid \ : \ u, v \in W\} \geq d_{GH}(\mathbb{P}^m(\varepsilon, x), [W, d^x, 0])$$

where $d^x(u,v) = \|u-v\|_x$. In the given regularity settings, we shall use the following expansion of d^x_ε: if u, v are linearly independent then

$$d^x_\varepsilon(u,v) \;=\; d^x(u,v) - \frac{1}{6}\varepsilon^2 K_x(u,v)\frac{\|u\|_x^2\|v\|_x^2 - \langle u,v\rangle_x^2}{d^x(u,v)} + \varepsilon^2\mathcal{O}(\varepsilon)$$

where K is the sectional curvature of the metric g. (If u, v are linearly dependent then $d^x_\varepsilon(u,v) = d^x(u,v)$.) From here we easily obtain that

$$\frac{1}{\varepsilon^2}d_{GH}(\mathbb{P}^m(\varepsilon,x),[T_xX,d^x,x]) \;\leq\; \frac{1}{3}\sup\{|\; K_x(u,v)\;|\;:\; u,v\in W \text{ lin. indep.}\} + \mathcal{O}(\varepsilon)$$

which ends the proof. $\square$

This proposition makes us define the "curvdimension" and "curvature" of an (abstract) metric profile.

Definition 2.17 *Let $\mathbb{P}$ be an abstract metric profile. The curvdimension of this abstract metric profile is*

$$curvdim\,\mathbb{P} \;=\; \sup\left(\left\{\alpha>0\;:\;\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^\alpha}d_{GH}(\mathbb{P}(\varepsilon),\mathbb{P}(0))=0\right\}\cup\{0\}\right) \tag{9}$$

and, in the case that the curvdimension equals $\beta>0$ then the β-curvature of $\mathbb{P}$ is the number $M>0$ such that

$$\lim_{\varepsilon\to 0}\log_\varepsilon\left(\frac{1}{M}d_{GH}(\mathbb{P}(\varepsilon),\mathbb{P}(0))\right)=\beta \tag{10}$$

In case $\mathbb{P}$ is the metric profile of a point $x\in X$ in a metric space (X,d) then the curvdimension at x and curvature at x are the curvimension, respectively curvature, of the metric profile of x.

It follows that non-flat riemannian (smooth enough) spaces have curvdimension 2. Also, any metric cone has curvdimension equal to 0 (meaning "all metric cones are flat objects"). In particular, finite dimensional normed vector spaces are flat (as they should be).

3 Groups with dilations

We shall see that for a dilation structure (or "dilatation structure", or "metric space with dilations") the metric tangent spaces have the structure of a normed local group with dilations. The notion has been introduced in [9], [5]; we describe it further.

We shall work with local groups. We start with the following setting (slightly non standard): G is a topological group endowed with an uniformity such that the operation is uniformly continuous.

We introduce first the double of G, as the group $G^{(2)} \;=\; G\times G$ with operation

$$(x,u)(y,v) \;=\; (xy, y^{-1}uyv) \quad .$$

The operation on the group G, seen as the function

$$op: G^{(2)}\to G\;,\quad op(x,y) \;=\; xy \quad ,$$

is a group morphism. Also the inclusions:

$$i': G\to G^{(2)}\;,\quad i'(x) \;=\; (x,e)$$

$$i": G\to G^{(2)}\;,\quad i"(x) \;=\; (x,x^{-1})$$

are group morphisms.

An uniform group is a group G such that left translations are uniformly continuous functions and the left action of G on itself is uniformly continuous too. In order to precisely formulate this we shall need two uniformities: one on G and another on $G \times G$. These uniformities should be compatible, which is achieved by saying that i', i" are uniformly continuous. The uniformity of the group operation is achieved by saying that the *op* morphism is uniformly continuous.

Definition 3.1 *1. G is an uniform group if we have two uniformity structures, on G and G^2, such that op, i', i" are uniformly continuous.*

2. A local action of a uniform group G on a uniform pointed space (X, x_0) is a function $\phi \in W \in \mathcal{V}(e) \mapsto \hat{\phi} : U_\phi \in \mathcal{V}(x_0) \to V_\phi \in \mathcal{V}(x_0)$ such that:

(a) the map $(\phi, x) \mapsto \hat{\phi}(x)$ is uniformly continuous from $G \times X$ (with product uniformity) to X,

(b) for any $\phi, \psi \in G$ there is $D \in \mathcal{V}(x_0)$ such that for any $x \in D$ $\phi\hat{\psi}^{-1}(x)$ and $\hat{\phi}(\hat{\psi}^{-1}(x))$ make sense and $\phi\hat{\psi}^{-1}(x) = \hat{\phi}(\hat{\psi}^{-1}(x))$.

3. Finally, a local group is an uniform space G with an operation defined in a neighbourhood of $(e, e) \subset G \times G$ which satisfies the uniform group axioms locally.

Remark that a local group acts locally at left (and also by conjugation) on itself.

Let Γ be a topological commutative group, endowed with a continuous morphism $|\cdot|: \Gamma \to (0, +\infty)$. For example Γ could be $(0, +\infty)$ with the operation of multiplication of positive real numbers and the said morphism could be the identity. Or Γ could be the set of complex numbers different from 0, with the operation of multiplication of complex numbers and morphism taken to be the modulus function. Also, Γ could be the set of integers with the operation of addition and the morphism could be the exponential function. Many other possibilities exist (like a product between a finite commutative group with one of the examples given before).

It is useful further to just think that Γ is like in the first example, because in these notes we are not going to use the structure of Γ in order to put more geometrical objects on the metric space (like we do, for example, in the paper [M. Buliga, Braided spaces with dilations and sub-riemannian symmetric spaces. in: Geometry. Exploratory Workshop on Differential Geometry and its Applications, eds. D. Andrica, S. Moroianu, Cluj-Napoca 2011, 21-35], arXiv:1005.5031).

The elements of Γ will be denoted with small greek letters, like $\varepsilon, \mu, ...$ By covention, whenever we write "$\varepsilon \to 0$", we really mean "$|\varepsilon| \to 0$". Also "$\mathcal{O}(\varepsilon)$" means "$\mathcal{O}(|\varepsilon|)$", and so on.

Definition 3.2 *A local group with dilations (G, δ) is a local group G with a local action of Γ (denoted by δ), on G such that*

H0. the limit $\lim_{\varepsilon \to 0} \delta_\varepsilon x = e$ exists and is uniform with respect to x in a compact neighbourhood of the identity e.

H1. the limit

$$\beta(x, y) = \lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \left((\delta_\varepsilon x)(\delta_\varepsilon y) \right)$$

is well defined in a compact neighbourhood of e and the limit is uniform.

H2. the following relation holds

$$\lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \left((\delta_\varepsilon x)^{-1} \right) = x^{-1}$$

where the limit from the left hand side exists in a neighbourhood of e and is uniform with respect to x.

Definition 3.3 *A normed local group with dilations $(G, \delta, \|\cdot\|)$ is a local group with dilations (G, δ) endowed with a continuous norm function $\|\cdot\| : G \to \mathbb{R}$ which satisfies (locally, in a neighbourhood of the neutral element e) the properties:*

(a) the limit $\lim_{\varepsilon\to 0} \frac{1}{\nu(\varepsilon)}\|\delta_\varepsilon x\| = \|x\|^N$ exists, is uniform with respect to x in compact set,

(b) if $\|x\|^N = 0$ then $x = e$.

In a normed local group with dilations we have a natural left invariant (locally defined) distance given by

$$d(x, y) = \|x^{-1}y\| \quad . \tag{11}$$

In a (local) group with dilations (G, δ) we define dilations based in any point $x \in G$ by

$$\delta_\varepsilon^x u = x\delta_\varepsilon(x^{-1}u). \tag{12}$$

3.1 Conical groups

Definition 3.4 *A normed conical group N is a normed group with dilations such that for any $\varepsilon \in \Gamma$ the dilation δ_ε is a group morphism and such that for any $\varepsilon > 0$ $\|\delta_\varepsilon x\| = \nu(\varepsilon)\|x\|$.*

A conical group is the infinitesimal version of a group with dilations ([5] proposition 2).

Proposition 3.5 *Under the hypotheses H0, H1, H2 $(G, \beta, \delta, \|\cdot\|^N)$ is a local normed conical group, with operation β, dilations δ and homogeneous norm $\|\cdot\|^N$.*

Proof. All the uniformity assumptions allow us to change at will the order of taking limits. We shall not insist on this further and we shall concentrate on the algebraic aspects.

We have to prove the associativity, existence of neutral element, existence of inverse and the property of being conical.

For the associativity $\beta(x, \beta(y, z)) = \beta(\beta(x, y), z)$ we compute:

$$\beta(x, \beta(y, z)) = \lim_{\varepsilon\to 0,\eta\to 0} \delta_\varepsilon^{-1}\left\{(\delta_\varepsilon x)\delta_{\varepsilon/\eta}\left((\delta_\eta y)(\delta_\eta z)\right)\right\} \quad .$$

We take $\varepsilon = \eta$ and we get

$$\beta(x, \beta(y, z)) = \lim_{\varepsilon\to 0}\left\{(\delta_\varepsilon x)(\delta_\varepsilon y)(\delta_\varepsilon z)\right\} \quad .$$

In the same way:

$$\beta(\beta(x, y), z) = \lim_{\varepsilon\to 0,\eta\to 0} \delta_\varepsilon^{-1}\left\{(\delta_{\varepsilon/\eta}x)\left((\delta_\eta x)(\delta_\eta y)\right)(\delta_\varepsilon z)\right\} \quad .$$

and again taking $\varepsilon = \eta$ we obtain

$$\beta(\beta(x, y), z) = \lim_{\varepsilon\to 0}\left\{(\delta_\varepsilon x)(\delta_\varepsilon y)(\delta_\varepsilon z)\right\} = \beta(x, \beta(y, z)) \quad .$$

The neutral element is e, from H0 (first part): $\beta(x, e) = \beta(e, x) = x$. The inverse of x is x^{-1}, by a similar argument:

$$\beta(x, x^{-1}) = \lim_{\varepsilon\to 0,\eta\to 0} \delta_\varepsilon^{-1}\left\{(\delta_\varepsilon x)\left(\delta_{\varepsilon/\eta}(\delta_\eta x)^{-1}\right)\right\} \quad ,$$

and taking $\varepsilon = \eta$ we obtain

$$\beta(x, x^{-1}) = \lim_{\varepsilon \to 0} \delta_\varepsilon^{-1} \left((\delta_\varepsilon x)(\delta_\varepsilon x)^{-1} \right) = \lim_{\varepsilon \to 0} \delta_\varepsilon^{-1}(e) = \ e \quad .$$

Finally, β has the property:

$$\beta(\delta_\eta x, \delta_\eta y) = \delta_\eta \beta(x, y) \quad ,$$

which comes from the definition of β and commutativity of multiplication in $(0, +\infty)$. This proves that (G, β, δ) is conical. □

3.2 Carnot groups

Carnot groups appear in sub-riemannian geometry as models of tangent spaces, [3], [22], [28]. In particular such groups can be endowed with a structure of sub-riemannian manifold.

Definition 3.6 *A Carnot (or stratified homogeneous) group is a pair (N, V_1) consisting of a real connected simply connected group N with a distinguished subspace V_1 of the Lie algebra $Lie(N)$, such that the following direct sum decomposition occurs:*

$$n \ = \ \sum_{i=1}^{m} V_i \ , \quad V_{i+1} \ = \ [V_1, V_i]$$

The number m is the step of the group. The number $Q \ = \ \sum_{i=1}^{m} i \ dim V_i$ is called the homogeneous dimension of the group.

Because the group is nilpotent and simply connected, the exponential mapping is a diffeomorphism. We shall identify the group with the algebra, if is not locally otherwise stated.

The structure that we obtain is a set N endowed with a Lie bracket and a group multiplication operation, obtained by the Baker-Campbell-Hausdorff formula. Remark that the group operation is polynomial, because the Baker-Campbell-Hausdorff formula contains only a finite number of terms, due to the fact that the Lie bracket is nilpotent.

Any Carnot group admits a one-parameter family of dilations. For any $\varepsilon > 0$, the associated dilation is:

$$x \ = \ \sum_{i=1}^{m} x_i \ \mapsto \ \delta_\varepsilon x \ = \ \sum_{i=1}^{m} \varepsilon^i x_i$$

Any such dilation is a group morphism and a Lie algebra morphism.

In a Carnot group N let us choose an euclidean norm $\| \cdot \|$ on V_1. We shall endow the group N with a structure of a sub-riemannian manifold. For this take the distribution obtained from left translates of the space V_1. The metric on that distribution is obtained by left translation of the inner product restricted to V_1.

Because V_1 generates (the algebra) N then any element $x \in N$ can be written as a product of elements from V_1, in a controlled way, described in the following useful lemma (slight reformulation of Lemma 1.40, Folland, Stein [19]).

Lemma 3.7 *Let N be a Carnot group and $X_1, ..., X_p$ an orthonormal basis for V_1. Then there is a a natural number M and a function $g : \{1, ..., M\} \to \{1, ..., p\}$ such that any element $x \in N$ can be written as:*

$$x \ = \ \prod_{i=1}^{M} \exp(t_i X_{g(i)}) \tag{13}$$

Moreover, if x is sufficiently close (in Euclidean norm) to 0 then each t_i can be chosen such that $\mid t_i \mid \leq C\|x\|^{1/m}$

As a consequence we get:

Corollary 3.8 *The Carnot-Carathéodory distance*

$$d(x,y) \;=\; \inf\left\{\int_0^1 \|c^{-1}\dot{c}\|\; dt \;\;:\;\; c(0)=x,\; c(1)=y,\right.$$

$$\left. c^{-1}(t)\dot{c}(t) \in V_1 \text{ for a.e. } t\in[0,1]\right\}$$

is finite for any two $x, y \in N$. The distance is obviously left invariant, thus it induces a norm on N.

The Carnot-Carathéodory distance induces a homogeneous norm on the Carnot group N by the formula: $\|x\| = d(0,x)$. From the invariance of the distance with respect to left translations we get: for any $x, y \in N$

$$\|x^{-1}y\| = d(x,y)$$

For any $x \in N$ and $\varepsilon > 0$ we define the dilation $\delta_\varepsilon^x y = x\delta_\varepsilon(x^{-1}y)$. Then (N, d, δ) is a dilation structure, according to theorem 5.1.

3.3 Contractible groups

Definition 3.9 *A contractible group is a pair (G,α), where G is a topological group with neutral element denoted by e, and $\alpha \in Aut(G)$ is an automorphism of G such that:*

- *α is continuous, with continuous inverse,*
- *for any $x \in G$ we have the limit $\lim\limits_{n\to\infty} \alpha^n(x) = e$.*

For a contractible group (G,α), the automorphism α is compactly contractive (Lemma 1.4 (iv) [29]), that is: for each compact set $K \subset G$ and open set $U \subset G$, with $e \in U$, there is $N(K,U) \in \mathbb{N}$ such that for any $x \in K$ and $n \in \mathbb{N}$, $n \geq N(K,U)$, we have $\alpha^n(x) \in U$.

If G is locally compact then α compactly contractive is equivalent with: each identity neighbourhood of G contains an α-invariant neighbourhood. Further on we shall assume without mentioning that all groups are locally compact.

Any conical group can be seen as a contractible group. Indeed, it suffices to associate to a conical group (G,δ) the contractible group (G,δ_ε), for a fixed $\varepsilon \in \Gamma$ with $\nu(\varepsilon) < 1$.

Conversely, to any contractible group (G,α) we may associate the conical group (G,δ), with $\Gamma = \left\{\dfrac{1}{2^n} \,:\, n \in \mathbb{N}\right\}$ and for any $n \in \mathbb{N}$ and $x \in G$

$$\delta_{\frac{1}{2^n}} x \;=\; \alpha^n(x) \quad .$$

(Finally, a local conical group has only locally the structure of a contractible group.)

The structure of contractible groups is known in some detail, due to Siebert [29], Wang [38], Glöckner and Willis [21], Glöckner [20] and references therein.

For this paper the following results are of interest. We begin with the definition of a contracting automorphism group [29], definition 5.1.

Definition 3.10 *Let G be a locally compact group. An automorphism group on G is a family $T = (\tau_t)_{t>0}$ in $Aut(G)$, such that $\tau_t \tau_s = \tau_{ts}$ for all $t, s > 0$.*

The contraction group of T is defined by

$$C(T) \;=\; \left\{ x \in G \,:\, \lim_{t \to 0} \tau_t(x) = e \right\} \quad .$$

The automorphism group T is contractive if $C(T) = G$.

It is obvious that a contractive automorphism group T induces on G a structure of conical group. Conversely, any conical group with $\Gamma = (0, +\infty)$ has an associated contractive automorphism group (the group of dilations based at the neutral element).

Further is proposition 5.4 [29].

Proposition 3.11 *For a locally compact group G the following assertions are equivalent:*

(i) G admits a contractive automorphism group;

(ii) G is a simply connected Lie group whose Lie algebra admits a positive graduation.

4 Dilation structures

In this paper I use the denomination "dilation structures", compared with older papers, where the name "dilatation structures" was used. Another name which could be used is "metric space with dilations".

We shall use here a slightly particular version of dilation structures. For the general definition of a dilation structure see [5] (the general definition applies for dilation structures over ultrametric spaces as well).

4.1 Normed groupoids with dilations

Notions of convergence. Any norm d on a groupoid G induces three notions of convergence on the set of arrows G.

Definition 4.1 *A net of arrows (a_ε)* **simply converges** *to the arrow $a \in G$ (we write $a_\varepsilon \to a$) if:*

(i) for any $\varepsilon \in I$ there are elements $g_\varepsilon, h_\varepsilon \in G$ such that $h_\varepsilon a_\varepsilon g_\varepsilon = a$,

(ii) we have $\lim\limits_{\varepsilon \in I} d(g_\varepsilon) = 0$ and $\lim\limits_{\varepsilon \in I} d(h_\varepsilon) = 0$.

A net of arrows (a_ε) **left-converges** *to the arrow $a \in G$ (we write $a_\varepsilon \overset{L}{\to} a$) if for all $i \in I$ we have $(a_\varepsilon^{-1}, a) \in G^{(2)}$ and moreover $\lim\limits_{\varepsilon \in I} d(a_\varepsilon^{-1} a) = 0$.*

A net of arrows (a_ε) **right-converges** *to the arrow $a \in G$ (we write $a_\varepsilon \overset{R}{\to} a$) if for all $i \in I$ we have $(a_\varepsilon, a^{-1}) \in G^{(2)}$ and moreover $\lim\limits_{\varepsilon \in I} d(a_\varepsilon a^{-1}) = 0$.*

It is clear that if $a_\varepsilon \overset{L}{\to} a$ or $a_\varepsilon \overset{R}{\to} a$ then $a_\varepsilon \to a$.

Right-convergence of a_ε to a is just convergence of a_ε to a in the distance $d_{\alpha(a)}$, that is $\lim\limits_{\varepsilon \in I} d_{\alpha(a)}(a_\varepsilon, a) = 0$.

Left-convergence of a_ε to a is just convergence of a_ε^{-1} to a^{-1} in the distance $d_{\omega(a)}$, that is $\lim\limits_{\varepsilon \in I} d_{\omega(a)}(a_\varepsilon^{-1}, a^{-1}) = 0$.

Proposition 4.2 *Let G be a groupoid with a norm d.*

(i) If $a_\varepsilon \overset{L}{\to} a$ and $a_\varepsilon \overset{L}{\to} b$ then $a = b$. If $a_\varepsilon \overset{R}{\to} a$ and $a_\varepsilon \overset{R}{\to} b$ then $a = b$.

(ii) The following are equivalent:

1. *G is a Hausdorff topological groupoid with respect to the topology induced by the simple convergence,*
2. *d is a separable norm,*
3. *for any net (a_ε), if $a_\varepsilon \to a$ and $a_\varepsilon \to b$ then $a = b$.*
4. *for any net (a_ε), if $a_\varepsilon \overset{R}{\to} a$ and $a_\varepsilon \overset{L}{\to} b$ then $a = b$.*

Proof. (i) We prove only the first part of the conclusion. We can write $b^{-1}a = b^{-1}a_\varepsilon a_\varepsilon^{-1}a$, therefore

$$d(b^{-1}a) \leq d(b^{-1}a_\varepsilon) + d(a_\varepsilon^{-1}a)$$

The right hand side of this inequality is arbitrarily small, so $d(b^{-1}a) = 0$, which implies $a = b$.

(ii) Remark that the structure maps are continuous with respect to the topology induced by the simple convergence. We need only to prove the uniqueness of limits.

3. $\Rightarrow$ 4. is trivial. In order to prove that 4.$\Rightarrow$ 3., consider an arbitrary net (a_ε) such that $a_\varepsilon \to a$ and $a_\varepsilon \to b$. This means that there exist nets $(g_\varepsilon), (g'_\varepsilon), (h_\varepsilon), (h'_\varepsilon)$ such that $h_\varepsilon a_\varepsilon g_\varepsilon = a$, $h'_\varepsilon a_\varepsilon g'_\varepsilon = b$ and $\lim\limits_{i\in I}(d(g_\varepsilon) + d(g'_\varepsilon) + d(h_\varepsilon) + d(h'_\varepsilon)) = 0$. Let $g"_\varepsilon = g_\varepsilon^{-1}g'_\varepsilon$ and $h"_\varepsilon = h'_\varepsilon h_\varepsilon^{-1}$. We have then $b = h"_\varepsilon a g"_\varepsilon$ and $\lim\limits_{i\in I}(d(g"_\varepsilon) + d(h"_\varepsilon)) = 0$. Then $h"_\varepsilon a \overset{L}{\to} b$ and $h"_\varepsilon a \overset{R}{\to} a$. We deduce that $a = b$.

1.$\Leftrightarrow$ 3. is trivial. So is 3. $\Rightarrow$ 2. We finish the proof by showing that 2. $\Rightarrow$ 3. By a reasoning made previously, it is enough to prove that: if $b = h_\varepsilon a g_\varepsilon$ and $\lim\limits_{i\in I}(d(g_\varepsilon) + d(h_\varepsilon)) = 0$ then $a = b$. Because d is separable it follows that $\alpha(a) = \alpha(b)$ and $\omega(a) = \omega(b)$. We have then $a^{-1}b = a^{-1}h_\varepsilon a g_\varepsilon$, therefore

$$d(a^{-1}b) \leq d(a^{-1}h_\varepsilon a) + d(g_\varepsilon)$$

The norm d induces a left invariant distance on the vertex group of all arrows g such that $\alpha(g) = \omega(g) = \alpha(a)$. This distance is obviously continuous with respect to the simple convergence in the group. The net $a^{-1}h_\varepsilon a$ simply converges to $\alpha(a)$ by the continuity of the multiplication (indeed, h_ε simply converges to $\alpha(a)$). Therefore $\lim\limits_{i\in I} d(a^{-1}h_\varepsilon a) = 0$. It follows that $d(a^{-1}b)$ is arbitrarily small, therefore $a = b$. □

By adapting the definition of a normed group with dilations to a normed groupoid with dilations, we get the following structure.

Definition 4.3 *A* **dilation of a separated normed groupoid** (G, d) *is a map assigning to any $\varepsilon \in \Gamma$ a transformation $\delta_\varepsilon : dom(\varepsilon) \to im(\varepsilon)$ which satisfies the following:*

A1. For any $\varepsilon \in \Gamma$ $\alpha\delta_\varepsilon = \alpha$. Moreover $\varepsilon \in \Gamma \mapsto \delta_\varepsilon$ is an action of Γ on G, that is for any $\varepsilon, \mu \in \Gamma$ we have $\delta_\varepsilon\delta_\mu = \delta_{\varepsilon\mu}$, $(\delta_\varepsilon)^{-1} = \delta_{\varepsilon^{-1}}$ and $\delta_e = id$.

A2. For any $x \in Ob(G)$ and any $\varepsilon \in \Gamma$ we have $\delta_\varepsilon(x) = x$. Moreover the transformation δ_ε contracts $dom(\varepsilon)$ to $X = Ob(G)$ uniformly on bounded sets, which means that the net $d\,\delta_\varepsilon$ converges to the constant function 0, uniformly on bounded sets.

A3. There is a function $\bar{d} : G \times_\alpha G \cap U^2 \to \mathbb{R}$ *which is the limit*

$$\lim_{\varepsilon \to 0} \frac{1}{\mid \varepsilon \mid} d\, dif(\delta_\varepsilon g, \delta_\varepsilon h) \;=\; \tilde{d}_0(g,h)$$

uniformly on bounded sets in the sense of definition 4.1. Moreover the convergence with respect to $\bar{d}$ *is the same as the convergence with respect to* $\tilde{d}$ *and in particular* $\tilde{d}_0(g,h) = 0$ *implies* $g = h$.

A4weak. There is a dilation $\bar{\delta}$ *of the normed groupoid* $(G \times_\alpha G, \tilde{d}_0)$ *such that for any* $\varepsilon \in \Gamma$ *the transformation* $\tilde{\delta}_{\mu,\varepsilon}$ *converges uniformly on bounded sets to* $\bar{\delta}_\varepsilon$.

The domains and codomains of a dilation of (G,d) *satisfy the following Axiom A0:*

(i) for any $\varepsilon \in \Gamma$ $Ob(G) = X \subset dom(\varepsilon)$ *and* $dom(\varepsilon) = dom(\varepsilon)^{-1}$,

(ii) for any bounded set $K \subset Ob(G)$ *there are* $1 < A < B$ *such that for any* $\varepsilon \in \Gamma$, $\mid \varepsilon \mid \leq 1$*:*

$$d^{-1}(\mid \varepsilon \mid) \cap \alpha^{-1}(K) \subset \delta_\varepsilon \left(d^{-1}(A) \cap \alpha^{-1}(K) \right) \subset dom(\varepsilon^{-1}) \cap \alpha^{-1}(K) \subset$$

$$\subset \delta_\varepsilon \left(d^{-1}(B) \cap \alpha^{-1}(K) \right) \subset \delta_\varepsilon \left(dom(\varepsilon) \cap \alpha^{-1}(K) \right) \tag{14}$$

(iii) for any bounded set $K \subset Ob(G)$ *there are* $R > 0$ *and* $\varepsilon_0 \in (0,1]$ *such that for any* $\varepsilon \in \Gamma$, $\mid \varepsilon \mid \leq \varepsilon_0$ *and any* $g, h \in \mid d^{-1}(R) \cap \alpha^{-1}(K)$ *we have:*

$$dif(\delta_\varepsilon g, \delta_\varepsilon h) \in dom(\varepsilon^{-1}) \tag{15}$$

4.2 Dilation structures, definition

Applied to the trivial normed groupoid associated to a metric space, this gives:

Definition 4.4 *Let* (X,d) *be a complete metric space such that for any* $x \in X$ *the closed ball* $\bar{B}(x,3)$ *is compact. A* **dilation structure** (X,d,δ) *over* (X,d) *is the assignment to any* $x \in X$ *and* $\varepsilon \in (0,+\infty)$ *of a invertible homeomorphism, defined as: if* $\varepsilon \in (0,1]$ *then* $\delta_\varepsilon^x : U(x) \to V_\varepsilon(x)$, *else* $\delta_\varepsilon^x : W_\varepsilon(x) \to U(x)$, *such that the following axioms are satisfied:*

A0. *For any* $x \in X$ *the sets* $U(x), V_\varepsilon(x), W_\varepsilon(x)$ *are open neighbourhoods of* x. *There are numbers* $1 < A < B$ *such that for any* $x \in X$ *and any* $\varepsilon \in (0,1)$ *we have the following string of inclusions:*

$$B_d(x,\varepsilon) \subset \delta_\varepsilon^x B_d(x,A) \subset V_\varepsilon(x) \subset W_{\varepsilon^{-1}}(x) \subset \delta_\varepsilon^x B_d(x,B)$$

Moreover for any compact set $K \subset X$ *there are* $R = R(K) > 0$ *and* $\varepsilon_0 = \varepsilon(K) \in (0,1)$ *such that for all* $u, v \in \bar{B}_d(x,R)$ *and all* $\varepsilon \in (0,\varepsilon_0)$, *we have*

$$\delta_\varepsilon^x v \in W_{\varepsilon^{-1}}(\delta_\varepsilon^x u) \ .$$

A1. *We have* $\delta_\varepsilon^x x = x$ *for any point* x. *We also have* $\delta_1^x = id$ *for any* $x \in X$. *Let us define the topological space*

$$dom\, \delta = \{(\varepsilon, x, y) \in (0,+\infty) \times X \times X \ : \ if\ \varepsilon \leq 1\ then\ y \in U(x) \ ,$$

$$else\ y \in W_\varepsilon(x)\}$$

with the topology inherited from $(0,+\infty) \times X \times X$ *endowed with the product topology. Consider also* $Cl(dom\, \delta)$, *the closure of* $dom\, \delta$ *in* $[0,+\infty) \times X \times X$. *The function* $\delta : dom\, \delta \to X$ *defined by* $\delta(\varepsilon, x, y) = \delta_\varepsilon^x y$ *is continuous. Moreover, it can be continuously extended to the set* $Cl(dom\, \delta)$ *and we have*

$$\lim_{\varepsilon \to 0} \delta_\varepsilon^x y \;=\; x$$

A2. *For any $x, \in X$, $\varepsilon, \mu \in (0, +\infty)$ and $u \in U(x)$ we have the equality:*

$$\delta_{\varepsilon}^{x}\delta_{\mu}^{x}u = \delta_{\varepsilon\mu}^{x}u$$

whenever one of the sides are well defined.

A3. *For any x there is a distance function $(u, v) \mapsto d^{x}(u, v)$, defined for any u, v in the closed ball (in distance d) $\bar{B}(x, A)$, such that*

$$\lim_{\varepsilon \to 0} \quad \sup \left\{ \mid \frac{1}{\varepsilon} d(\delta_{\varepsilon}^{x}u, \delta_{\varepsilon}^{x}v) \; - \; d^{x}(u, v) \mid \; : u, v \in \bar{B}_{d}(x, A) \right\} \; = \; 0$$

uniformly with respect to x in compact set.

The **dilation structure is strong** *if it satisfies the following supplementary condition:*

A4. *Let us define $\Delta_{\varepsilon}^{x}(u, v) = \delta_{\varepsilon^{-1}}^{\delta_{\varepsilon}^{x}u}\delta_{\varepsilon}^{x}v$. Then we have the limit*

$$\lim_{\varepsilon \to 0} \Delta_{\varepsilon}^{x}(u, v) = \Delta^{x}(u, v)$$

uniformly with respect to x, u, v in compact set.

We shall use many times from now the words "sufficiently close". This deserves a definition.

Definition 4.5 *Let (X, d, δ) be a strong dilation structure. We say that a property*

$$\mathcal{P}(x_1, x_2, x_3, ...)$$

is true for $x_1, x_2, x_3, ...$ **sufficiently close** *if for any compact, non empty set $K \subset X$, there is a positive constant $C(K) > 0$ such that $\mathcal{P}(x_1, x_2, x_3, ...)$ is true for any $x_1, x_2, x_3, ... \in K$ with $d(x_i, x_j) \leq C(K)$.*

5 Some examples of dilation structures

5.1 Snowflakes

The next example is a snowflake variation of the euclidean case: $X = \mathbb{R}^n$ and for any $a \in (0, 1]$ take

$$d_a(x, y) \; = \; \|x - y\|^{\alpha} \quad , \quad \delta_{\varepsilon}^{x}y \; = \; x + \varepsilon^{\frac{1}{a}}(y - x) \; .$$

We leave to the reader to verify the axioms.

More general, if (X, d, δ) is a dilation structure then $(X, d_a, \delta(a))$ is also a dilation structure, for any $a \in (0, 1]$, where

$$d_a(x, y) \; = \; (d(x, y))^{a} \quad , \quad \delta(a)_{\varepsilon}^{x} \; = \; \delta_{\varepsilon^{\frac{1}{a}}}^{x} \; .$$

5.2 Nonstandard dilations in the euclidean space

Take $X = \mathbb{R}^2$ with the euclidean distance. For any $z \in \mathbb{C}$ of the form $z = 1 + i\theta$ we define dilations

$$\delta_\varepsilon x = \varepsilon^z x \ .$$

It is easy to check that $(X, \delta, +, d)$ is a conical group, equivalenty that the dilations

$$\delta_\varepsilon^x y = x + \delta_\varepsilon (y - x) \ .$$

form a dilation structure with the euclidean distance.

Two such dilation structures (constructed with the help of complex numbers $1 + i\theta$ and $1 + i\theta'$) are equivalent if and only if $\theta = \theta'$.

There are two other surprising properties of these dilation structures. The first is that if $\theta \neq 0$ then there are no non trivial Lipschitz curves in X which are differentiable almost everywhere. The second property is that any holomorphic and Lipschitz function from X to X (holomorphic in the usual sense on $X = \mathbb{R}^2 = \mathbb{C}$) is differentiable almost everywhere, but there are Lipschitz functions from X to X which are not differentiable almost everywhere (suffices to take a $\mathcal{C}^\infty$ function from $\mathbb{R}^2$ to $\mathbb{R}^2$ which is not holomorphic).

5.3 Normed groups with dilations

The following result is theorem 15 [5].

Theorem 5.1 *Let $(G, \delta, \|\cdot\|)$ be a locally compact normed local group with dilations. Then (G, d, δ) is a dilation structure, where δ are the dilations defined by (12) and the distance d is induced by the norm as in (11).*

Proof. The axiom A0 is straightforward from definition 3.1, definition 3.2, axiom H0, and because the dilation structure is left invariant, in the sense that the transport by left translations in G preserves the dilations δ. We also trivially have axioms A1 and A2 satisfied.

For the axiom A3 remark that

$$d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) = d(x\delta_\varepsilon(x^{-1}u), x\delta_\varepsilon(x^{-1}u)) = d(\delta_\varepsilon(x^{-1}u), \delta_\varepsilon(x^{-1}v)).$$

Denote $U = x^{-1}u$, $V = x^{-1}v$ and for $\varepsilon > 0$ let

$$\beta_\varepsilon(u, v) = \delta_\varepsilon^{-1}\left((\delta_\varepsilon u)(\delta_\varepsilon v)\right).$$

We have then:

$$\frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) = \frac{1}{\varepsilon} \|\delta_\varepsilon \beta_\varepsilon \left(\delta_\varepsilon^{-1}\left((\delta_\varepsilon V)^{-1}\right), U\right)\| \quad .$$

Define the function

$$d^x(u, v) = \|\beta(V^{-1}, U)\|^N.$$

From definition 3.2 axioms H1, H2, and from definition 3.3 (d), we obtain that axiom A3 is satisfied.

For the axiom A4 we have to compute:

$$\Delta^x(u, v) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v = \left(\delta_\varepsilon^x u\right)\left(\delta_\varepsilon\right)^{-1}\left(\left(\delta_\varepsilon^x u\right)^{-1}\left(\delta_\varepsilon^x v\right)\right) =$$

$$= \left(x\delta_\varepsilon U\right)\beta_\varepsilon\left(\delta_\varepsilon^{-1}\left(\left(\delta_\varepsilon V\right)^{-1}\right), U\right) \to x\beta\left(V^{-1}, U\right)$$

as $\varepsilon \to 0$. Therefore axiom A4 is satisfied. □

5.4 Riemannian manifolds

The following interesting quotation from Gromov book [23], pages 85-86, motivates some of the ideas underlying dilation structures, especially in the very particular case of a riemannian manifold:

"**3.15. Proposition:** *Let (V, g) be a Riemannian manifold with g continuous. For each $v \in V$ the spaces $(V, \lambda d, v)$ Lipschitz converge as $\lambda \to \infty$ to the tangent space $(T_v V, 0)$ with its Euclidean metric g_v.*

Proof$_+$: *Start with a C^1 map $(\mathbb{R}^n, 0) \to (V, v)$ whose differential is isometric at 0. The λ-scalings of this provide almost isometries between large balls in $\mathbb{R}^n$ and those in λV for $\lambda \to \infty$.* **Remark:** *In fact we can define Riemannian manifolds as locally compact path metric spaces that satisfy the conclusion of Proposition 3.15.*"

The problem of domains and codomains left aside, any chart of a Riemannian manifold induces locally a dilation structure on the manifold. Indeed, take (M, d) to be a n-dimensional Riemannian manifold with d the distance on M induced by the Riemannian structure. Consider a diffeomorphism ϕ of an open set $U \subset M$ onto $V \subset \mathbb{R}^n$ and transport the dilations from V to U (equivalently, transport the distance d from U to V). There is only one thing to check in order to see that we got a dilation structure: the axiom A3, expressing the compatibility of the distance d with the dilations. But this is just a metric way to express the distance on the tangent space of M at x as a limit of rescaled distances (see Gromov Proposition 3.15, [23], p. 85-86). Denoting by g_x the metric tensor at $x \in U$, we have:

$$[d^x(u,v)]^2 =$$

$$= g_x \left(\frac{d}{d\,\varepsilon}_{|_{\varepsilon=0}} \phi^{-1}\left(\phi(x) + \varepsilon(\phi(u) - \phi(x))\right), \frac{d}{d\,\varepsilon}_{|_{\varepsilon=0}} \phi^{-1}\left(\phi(x) + \varepsilon(\phi(v) - \phi(x))\right) \right)$$

A basically different example of a dilation structure on a riemannian manifold will be explained next. Let M be a n dimensional riemannian manifold and exp be the geodesic exponential. To any point $x \in M$ and any vector $v \in T_x M$ the point $\exp_x(v) \in M$ is located on the geodesic passing thru x and tangent to v; if we parameterize this geodesic with respect to length, such that the tangent at x is parallel and has the same direction as v, then $\exp_x(v) \in M$ has the coordinate equal with the length of v with respect to the norm on $T_x M$. We define implicitly the dilation based at x, of coefficient $\varepsilon > 0$ by the relation:

$$\delta^x_\varepsilon \exp_x(u) \;=\; \exp_x(\varepsilon u) \quad .$$

It is not straightforward to check that we obtain a strong dilation structure, but it is true. There are interesting facts related to the numbers A, B and the minimal regularity required for the riemannian manifold. This example is different from the first because instead of using a chart (same for all x) we use a family of charts indexed with respect to the basepoint of the dilations.

6 Length dilation structures

6.1 Length in metric spaces

For a detailed introduction into the subject see for example [1], chapter 1.

Definition 6.1 *The* **(upper) dilation of a map** $f : X \to Y$ *between metric spaces, in a point* $u \in Y$ *is*

$$Lip(f)(u) = \limsup_{\varepsilon \to 0} \; \sup \left\{ \frac{d_Y(f(v), f(w))}{d_X(v,w)} \;\; : \;\; v \neq w \;,\; v, w \in B(u, \varepsilon) \right\}$$

In the particular case of a derivable function $f : \mathbb{R} \to \mathbb{R}^n$ the upper dilation is $Lip(f)(t) = \|\dot{f}(t)\|$.

A function $f : (X, d) \to (Y, d')$ is Lipschitz if there is a positive constant C such that for any $x, y \in X$ we have $d'(f(x), f(y)) \leq C\, d(x, y)$. The number $Lip(f)$ is the smallest such positive constant. Then for any $x \in X$ we have the obvious relation $Lip(f)(x) \ \leq \ Lip(f)$.

A curve is a continuous function $c : [a, b] \to X$. The image of a curve is called path. Length measures paths. Therefore length does not depends on the reparameterization of the path and it is additive with respect to concatenation of paths.

Definition 6.2 *In a metric space (X, d) there are several ways to define the length:*

(a) The **length of a curve with L^1 upper dilation** *$c : [a, b] \to X$ is*

$$L(f) = \int_a^b Lip(c)(t)\ dt$$

(b) The **variation of a curve** *$c : [a, b] \to X$ is the quantity $Var(c) =$*

$$= \sup \left\{ \sum_{i=0}^{n} d(c(t_i), c(t_{i+1})) \ \ : a = t_0 < t_1 < ... < t_n < t_{n+1} = b \right\}$$

(c) The **length of the path** *$A = c([a, b])$ is the one-dimensional Hausdorff measure of the path.:*

$$l(A) \ = \ \lim_{\delta \to 0} \inf \left\{ \sum_{i \in I} diam\ E_i \ : \ diam\ E_i < \delta \ , \quad A \subset \bigcup_{i \in I} E_i \right\}$$

The definitions are not equivalent. For Lipschitz curves the first two definitions agree. For simple Lipschitz curves all definitions agree.

Theorem 6.3 *For each Lipschitz curve $c : [a, b] \to X$, we have*

$$L(c) \ = \ Var(c) \ \geq \ \mathcal{H}^1(c([a, b]))$$

If c is moreover injective then $\mathcal{H}^1(c([a, b])) \ = \ Var(f)$.

An important tool used in the proof of the previous theorem is the geometrically obvious, but not straightforward to prove in this generality, reparametrisation Theorem.

Theorem 6.4 *Any Lipschitz curve c admits a reparametrisation c' such that $Lip(c')(t) = 1$ for almost any $t \in [a, b]$.*

Definition 6.5 *We shall denote by l_d the* **length functional induced by the distance** *d, defined only on the family of Lipschitz curves. If the metric space (X, d) is connected by Lipschitz curves, then the length induces a new distance d_l, given by:*

$$d_l(x, y) \ = \ \inf \ \{ l_d(c([a, b])) \ : \ c : [a, b] \to X \quad Lipschitz \ ,$$

$$c(a) = x \ , \ c(b) = y \}$$

A **length metric space** *is a metric space (X, d), connected by Lipschitz curves, such that $d = d_l$.*

From theorem 6.3 we deduce that Lipschitz curves in complete length metric spaces are absolutely continuous. Indeed, here is the definition of an absolutely continuous curve (definition 1.1.1, chapter 1, [1]).

Definition 6.6 *Let (X,d) be a complete metric space. A curve $c:(a,b)\to X$ is* **absolutely continuous** *if there exists $m\in L^1((a,b))$ such that for any $a<s\leq t<b$ we have*

$$d(c(s),c(t))\leq \int_s^t m(r)\ dr.$$

Such a function m is called a **upper gradient** *of the curve c.*

According to theorem 6.3, for a Lipschitz curve $c:[a,b]\to X$ in a complete length metric space such a function $m\in L^1((a,b))$ is the upper dilation $Lip(c)$. More can be said about the expression of the upper dilation. We need first to introduce the notion of metric derivative of a Lipschitz curve.

Definition 6.7 *A curve $c:(a,b)\to X$ is* **metrically derivable** *in $t\in(a,b)$ if the limit*

$$md(c)(t)=\lim_{s\to t}\frac{d(c(s),c(t))}{\mid s-t\mid}$$

exists and it is finite. In this case $md(c)(t)$ is called the **metric derivative** *of c in t.*

For the proof of the following theorem see [1], theorem 1.1.2, chapter 1.

Theorem 6.8 *Let (X,d) be a complete metric space and $c:(a,b)\to X$ be an absolutely continuous curve. Then c is metrically derivable for $\mathcal{L}^1$-a.e. $t\in(a,b)$. Moreover the function $md(c)$ belongs to $L^1((a,b))$ and it is minimal in the following sense: $md(c)(t)\leq m(t)$ for $\mathcal{L}^1$-a.e. $t\in(a,b)$, for each upper gradient m of the curve c.*

Consider (X,d) a complete, locally compact metric space, and a triple (X,d,δ) which satisfies A0, A1, A2. Denote by $Lip([0,1],X,d)$ the space of d-Lipschitz curves $c:[0,1]\to X$. Let also l_d denote the length functional associated to the distance d.

6.2 Gamma-convergence of length functionals

Definition 6.9 *For any $\varepsilon\in(0,1)$ we define the* **length functional**

$$l_\varepsilon:\mathcal{L}_\varepsilon(X,d,\delta)\to[0,+\infty]\quad,\quad l_\varepsilon(x,c)\ =\ l^x_\varepsilon(c)\ =\ \frac{1}{\varepsilon}\,l_d(\delta^x_\varepsilon c)$$

The domain of definition of the functional l_ε is the space:

$$\mathcal{L}_\varepsilon(X,d,\delta)\ =\ \{(x,c)\in X\times\mathcal{C}([0,1],X)\ :\ c:[0,1]\in U(x)\ ,$$
$$\delta^x_\varepsilon c\ \textit{is}\ d-Lip\ \textit{and}\ Lip(\delta^x_\varepsilon c)\ \leq\ 2\,l_d(\delta^x_\varepsilon c)\}$$

The last condition from the definition of $\mathcal{L}_\varepsilon(X,d,\delta)$ is a selection of parameterization of the path $c([0,1])$. Indeed, by the reparameterization theorem, if $\delta^x_\varepsilon c:[0,1]\to(X,d)$ is a d-Lipschitz curve of length $L=l_d(\delta^x_\varepsilon c)$ then $\delta^x_\varepsilon c([0,1])$ can be reparameterized by length, that is there exists a increasing function $\phi:[0,L]\to[0,1]$ such that $c'=\delta^x_\varepsilon c\circ\phi$ is a d-Lipschitz curve with $Lip(c')\leq 1$. But we can use a second affine reparameterization which sends $[0,L]$ back to $[0,1]$ and we get a Lipschitz curve $c"$ with $c"([0,1])=c'([0,1])$ and $Lip(c")\leq 2l_d(c)$.

We shall use the following definition of Gamma-convergence (see the book [17] for the notion of Gamma-convergence). Notice the use of convergence of sequences only in the second part of the definition.

Definition 6.10 *Let Z be a metric space with distance function D and $(l_\varepsilon)_{\varepsilon>0}$ be a family of functionals $l_\varepsilon : Z_\varepsilon \subset Z \to [0,+\infty]$. Then l_ε* **Gamma-converges** *to the functional $l : Z_0 \subset Z \to [0,+\infty]$ if:*

*(a) (***liminf inequality***) for any function $\varepsilon \in (0,\infty) \mapsto x_\varepsilon \in Z_\varepsilon$ such that $\lim_{\varepsilon\to 0} x_\varepsilon = x_0 \in Z_0$ we have*

$$l(x_0) \leq \liminf_{\varepsilon\to 0} l_\varepsilon(x_\varepsilon)$$

*(b) (***existence of a recovery sequence***) For any $x_0 \in Z_0$ and for any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \varepsilon_n = 0$ there is a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in Z_{\varepsilon_n}$ for any $n \in \mathbb{N}$, such that*

$$l(x_0) = \lim_{n\to\infty} l_{\varepsilon_n}(x_n)$$

We shall take as the metric space Z the space $X \times \mathcal{C}([0,1],X)$ with the distance

$$D((x,c),(x',c')) = \max\{d(x,x')\,,\,\sup\{d(c(t),c'(t)) : t \in [0,1]\}\}$$

Let $\mathcal{L}(X,d,\delta)$be the class of all $(x,c) \in X \times \mathcal{C}([0,1],X)$ which appear as limits $(x_n,c_n) \to (x,c)$, with $(x_n,c_n) \in \mathcal{L}_{\varepsilon_n}(X,d,\delta)$, the family $(c_n)_n$ is d-equicontinuous and $\varepsilon_n \to 0$ as $n \to \infty$.

Definition 6.11 *A triple (X,d,δ) is a* **length dilation structure** *if (X,d) is a complete, locally compact metric space such that A0, A1, A2, are satisfied, together with the following axioms:*

A3L. *there is a functional $l : \mathcal{L}(X,d,\delta) \to [0,+\infty]$ such that for any $\varepsilon_n \to 0$ as $n \to \infty$ the sequence of functionals l_{ε_n} Gamma-converges to the functional l.*

A4+ *Let us define $\Delta^x_\varepsilon(u,v) = \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}}\delta^x_\varepsilon v$ and $\Sigma^x_\varepsilon(u,v) = \delta^x_{\varepsilon^{-1}}\delta^{\delta^x_\varepsilon u}_\varepsilon v$. Then we have the limits*

$$\lim_{\varepsilon\to 0} \Delta^x_\varepsilon(u,v) = \Delta^x(u,v)$$

$$\lim_{\varepsilon\to 0} \Sigma^x_\varepsilon(u,v) = \Sigma^x(u,v)$$

uniformly with respect to x,u,v in compact set.

Remark 6.12 *For strong dilation structures the axioms A0 - A4 imply A4+, cf. corollary 9 [5]. The transformations $\Sigma^x_\varepsilon(u,\cdot)$ have the interpretation of approximate left translations in the tangent space of (X,d) at x.*

For any $\varepsilon \in (0,1)$ and any $x \in X$ the length functional l^x_ε induces a distance on $U(x)$:

$$\mathring{d}^x_\varepsilon(u,v) = \inf\{l^x_\varepsilon(c) : (x,c) \in \mathcal{L}_\varepsilon(X,d,\delta)\,,\, c(0) = u\,,\, c(1) = v\}$$

In the same way the length functional l from A3L induces a distance $\mathring{d}^x$ on $U(x)$.

Gamma-convergence implies that

$$\mathring{d}^x(u,v) \geq \limsup_{\varepsilon\to 0} \mathring{d}^x_\varepsilon(u,v) \tag{16}$$

Remark 6.13 *Without supplementary hypotheses we cannot prove A3 from A3L, that is in principle length dilation structures are not strong dilation structures.*

7 Properties of (length) dilation structures

7.1 Metric profiles associated with dilation structures

In this subsection we shall look at dilation structures from the metric point of view, by using Gromov-Hausdorff distance and metric profiles.

We state the interpretation of the axiom A3 as a theorem. But before a definition: we denote by (δ, ε) the distance on

$$\bar{B}_{d^x}(x,1) \ = \ \{y \in X \, : \, d^x(x,y) \leq 1\}$$

given by

$$(\delta,\varepsilon)(u,v) \ = \ \frac{1}{\varepsilon} d(\delta^x_\varepsilon u, \delta^x_\varepsilon v) \ .$$

Theorem 7.1 *Let (X,d,δ) be a dilation structure. The following are consequences of axioms A0, ... , A3 only:*

(a) for all $u,v \in X$ such that $d(x,u) \leq 1$ and $d(x,v) \leq 1$ and all $\mu \in (0,A)$ we have:

$$d^x(u,v) \ = \ \frac{1}{\mu} d^x(\delta^x_\mu u, \delta^x_\mu v) \ .$$

We shall say that d^x has the cone property with respect to dilations.

(b) The curve $\varepsilon > 0 \mapsto \mathbb{P}^x(\varepsilon) \ = \ [\bar{B}_{d^x}(x,1), (\delta,\varepsilon), x]$ is a metric profile.

Proof. (a) Indeed, for $\varepsilon, \mu \in (0,1)$ we have:

$$| \ \frac{1}{\varepsilon\mu} d(\delta^x_\varepsilon \delta^x_\mu u, \delta^x_\varepsilon \delta^x_\mu v) \ - \ d^x(u,v) \ | \leq | \ \frac{1}{\varepsilon\mu} d(\delta^x_{\varepsilon\mu} u, \delta^x_\varepsilon \delta^x_\mu u) - \frac{1}{\varepsilon\mu} d(\delta^x_{\varepsilon\mu} v, \delta^x_\varepsilon \delta^x_\mu v) \ | +$$

$$+ \ | \ \frac{1}{\varepsilon\mu} d(\delta^x_{\varepsilon\mu} u, \delta^x_{\varepsilon\mu} v) \ - \ d^x(u,v) \ | \ .$$

Use now the axioms A2 and A3 and pass to the limit with $\varepsilon \to 0$. This gives the desired equality.

(b)We have to prove that $\mathbb{P}^x$ is a metric profile. For this we have to compare two pointed metric spaces:

$$\big((\delta^x, \varepsilon\mu), \bar{B}_{d^x}(x,1), x\big) \quad \text{and} \quad \left(\frac{1}{\mu}(\delta^x,\varepsilon), \bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,1), x\right) \ .$$

Let $u \in X$ such that

$$\frac{1}{\mu}(\delta^x,\varepsilon)(x,u) \leq 1 \ .$$

This means that:

$$\frac{1}{\varepsilon} d(\delta^x_\varepsilon x, \delta^x_\varepsilon u) \ \leq \ \mu \ .$$

Use further axioms A1, A2 and the cone property proved before:

$$\frac{1}{\varepsilon} d^x(\delta^x_\varepsilon x, \delta^x_\varepsilon u) \ \leq \ (\mathcal{O}(\varepsilon) + 1)\mu$$

therefore

$$d^x(x,u) \ \leq \ (\mathcal{O}(\varepsilon) + 1)\mu \ .$$

It follows that for any $u \in \bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,1)$ we can choose $w(u) \in \bar{B}_{d^x}(x,1)$ such that

$$\frac{1}{\mu} d^x(u, \delta^x_\mu w(u)) \;=\; \mathcal{O}(\varepsilon) \;.$$

We want to prove that

$$|\; \frac{1}{\mu}(\delta^x,\varepsilon)(u_1,u_2) \;-\; (\delta^x,\varepsilon\mu)(w(u_1),w(u_2))\;| \leq\; \mathcal{O}(\varepsilon\mu) + \frac{1}{\mu}\mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon)\;.$$

This goes as following:

$$\begin{aligned}
&|\; \frac{1}{\mu}(\delta^x,\varepsilon)(u_1,u_2) \;-\; (\delta^x,\varepsilon\mu)(w(u_1),w(u_2))\;| \\
&= |\; \frac{1}{\varepsilon\mu} d(\delta^x_\varepsilon u_1, \delta^x_\varepsilon u_2) \;-\; \frac{1}{\varepsilon\mu} d(\delta^x_{\varepsilon\mu} w(u_1), \delta^x_{\varepsilon\mu} w(u_2))\;| \leq \\
&\leq\; \mathcal{O}(\varepsilon\mu) \;+\; |\; \frac{1}{\varepsilon\mu} d(\delta^x_\varepsilon u_1, \delta^x_\varepsilon u_2) \;-\; \frac{1}{\varepsilon\mu} d(\delta^x_\varepsilon \delta^x_\mu w(u_1), \delta^x_\varepsilon \delta^x_\mu w(u_2))\;| \leq \\
&\leq\; \mathcal{O}(\varepsilon\mu) \;+\; \frac{1}{\mu}\mathcal{O}(\varepsilon) \;+\; \frac{1}{\mu}\,|\; d^x(u_1,u_2) \;-\; d^x(\delta^x_\mu w(u_1), \delta^x_\mu w(u_2))\;.
\end{aligned}$$

In order to obtain the last estimate we used twice axiom A3. We continue:

$$\begin{aligned}
&\mathcal{O}(\varepsilon\mu) \;+\; \frac{1}{\mu}\mathcal{O}(\varepsilon) \;+\; \frac{1}{\mu}\,|\; d^x(u_1,u_2) \;-\; d^x(\delta^x_\mu w(u_1), \delta^x_\mu w(u_2)) \;\leq \\
&\leq \mathcal{O}(\varepsilon\mu) \;+\; \frac{1}{\mu}\mathcal{O}(\varepsilon) \;+\; \frac{1}{\mu} d^x(u_1, \delta^x_\mu w(u_1)) \;+\; \frac{1}{\mu} d^x(u_1, \delta^x_\mu w(u_2)) \;\leq \\
&\leq\; \mathcal{O}(\varepsilon\mu) + \frac{1}{\mu}\mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon)\;.
\end{aligned}$$

This shows that the property (b) of a metric profile is satisfied. The property (a) is proved in theorem 7.2. □

The following theorem is related to Mitchell [27] theorem 1, concerning sub-riemannian geometry.

Theorem 7.2 *In the hypothesis of theorem 7.1, we have the following limit:*

$$\lim_{\varepsilon\to 0}\; \frac{1}{\varepsilon} \sup\{|\; d(u,v) - d^x(u,v)\;|\;\; :\; d(x,u) \leq \varepsilon\;,\; d(x,v) \leq \varepsilon\} \;=\; 0\;.$$

Therefore if d^x is a true (i.e. nondegenerate) distance, then (X,d) admits a metric tangent space in x.

Moreover, the curve $\varepsilon \mapsto [\bar{B}_{d^x}(x,1),(\delta,\varepsilon),x]$ has the following property: let $c \in (0,1)$. Then we have the inclusion:

$$\delta^x_{\mu^{-1}}\left(\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c)\right) \;\subset\; \bar{B}_{d^x}(x,1) \quad .$$

Moreover, the following Gromov-Hausdorff distance is of order $\mathcal{O}(\varepsilon)$ for μ fixed (that is the modulus of convergence $\mathcal{O}(\varepsilon)$ does not depend on μ) :

$$\mu\; d_{GH}\left([\bar{B}_{d^x}(x,1),(\delta^x,\varepsilon),x],\; [\delta^x_{\mu^{-1}}\left(\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c)\right),(\delta^x,\varepsilon\mu),x]\right) = \mathcal{O}(\varepsilon)\;.$$

For another Gromov-Hausdorff distance we have the estimate:

$$d_{GH}\left([\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c), \frac{1}{\mu}(\delta^x,\varepsilon), x]\;,\; [\delta^x_{\mu^{-1}}\left(\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c)\right),(\delta^x,\varepsilon\mu),x]\right) = \mathcal{O}(\varepsilon\mu)$$

when $\varepsilon \in (0,\varepsilon(c))$.

Proof. We start from the axioms A0, A3 and we use the cone property. By A0, for $\varepsilon \in (0,1)$ and $u, v \in \bar{B}_d(x, \varepsilon)$ there exist $U, V \in \bar{B}_d(x, A)$ such that

$$u = \delta_\varepsilon^x U, v = \delta_\varepsilon^x V.$$

By the cone property we have

$$\frac{1}{\varepsilon} \mid d(u,v) - d^x(u,v) \mid = \mid \frac{1}{\varepsilon} d(\delta_\varepsilon^x U, \delta_\varepsilon^x V) - d^x(U,V) \mid .$$

By A2 we have

$$\mid \frac{1}{\varepsilon} d(\delta_\varepsilon^x U, \delta_\varepsilon^x V) - d^x(U,V) \mid \leq \mathcal{O}(\varepsilon).$$

This proves the first part of the theorem.

For the second part of the theorem take any $u \in \bar{B}_{\frac{1}{\mu}(\delta^x, \varepsilon)}(x, c)$. We have then

$$d^x(x,u) \ \leq \ c\mu + \mathcal{O}(\varepsilon) \ .$$

Then there exists $\varepsilon(c) > 0$ such that for any $\varepsilon \in (0, \varepsilon(c))$ and u in the mentioned ball we have:

$$d^x(x,u) \leq \ \mu$$

In this case we can take directly $w(u) \ = \ \delta^x_{\mu^{-1}} u$ and simplify the string of inequalities from the proof of theorem 7.1, point (b), to get eventually the three points from the second part of the theorem. □

7.2 Infinitesimal translations

The following proposition contains the main relations between the difference, sum and inverse gates. In [5] I explained these relations as appearing from the equivalent formalism using binary decorated trees.

Proposition 7.3 *Let $(X, \circ_\varepsilon)_{\varepsilon \in \Gamma}$ be a Γ-irq. Then we have the relations:*

(a) $\Delta_\varepsilon^x(u, \Sigma_\varepsilon^x(u,v)) = v$ *(difference is the inverse of sum)*

(b) $\Sigma_\varepsilon^x(u, \Delta^{x_\varepsilon}(u,v)) = v$ *(sum is the inverse of difference)*

(c) $\Delta_\varepsilon^x(u,v) = \Sigma_\varepsilon^{x \circ_\varepsilon u}(inv_\varepsilon^x u, v)$ *(difference approximately equals the sum of the inverse)*

(d) $inv_\varepsilon^{x \circ u}\, inv_\varepsilon^x\, u = u$ *(inverse operation is approximatively an involution)*

(e) $\Sigma_\varepsilon^x(u, \Sigma_\varepsilon^{x \circ_\varepsilon u}(v,w)) = \Sigma_\varepsilon^x(\Sigma_\varepsilon^x(u,v), w)$ *(approximate associativity of the sum)*

(f) $inv_\varepsilon^x\, u = \Delta_\varepsilon^x(u,x)$

(g) $\Sigma_\varepsilon^x(x,u) = u$ *(neutral element at right).*

Theorem 7.4 *Let (X, d, δ) be a dilation structure. Then the "infinitesimal translations"*

$$L_u^x(v) = \lim_{\varepsilon \to 0} \Delta_\varepsilon^x(u,v)$$

are d^x isometries.

Proof. The first part of the conclusion of theorem 7.2 can be written as:

$$\sup\left\{\frac{1}{\varepsilon}\mid d(u,v)\ -\ d^x(u,v)\mid\ :\ d(x,u)\leq\frac{3}{2}\varepsilon,\ d(x,v)\leq\frac{3}{2}\varepsilon\right\}\to 0 \tag{17}$$

as $\varepsilon\to 0$.

For $\varepsilon>0$ sufficiently small the points $x,\delta_\varepsilon^x u,\delta_\varepsilon^x v,\delta_\varepsilon^x w$ are close one to another. Precisely, we have

$$d(\delta_x^\varepsilon u,\delta_x^\varepsilon v)=\varepsilon(d^x(u,v)+\mathcal{O}(\varepsilon))\quad .$$

Therefore, if we choose u,v,w such that $d^x(u,v)<1, d^x(u,w)<1$, then there is $\eta>0$ such that for all $\varepsilon\in(0,\eta)$ we have

$$d(\delta_x^\varepsilon u,\delta_x^\varepsilon v)\leq\frac{3}{2}\varepsilon\quad ,\quad d(\delta_x^\varepsilon u,\delta_x^\varepsilon v)\leq\frac{3}{2}\varepsilon\quad .$$

We apply the estimate (17) for the basepoint $\delta_\varepsilon^x u$ to get:

$$\frac{1}{\varepsilon}\mid d(\delta_\varepsilon^x v,\delta_\varepsilon^x w)\ -\ d^{\delta_\varepsilon^x u}(\delta_\varepsilon^x v,\delta_\varepsilon^x w)\mid\to 0$$

when $\varepsilon\to 0$. This can be written, using the cone property of the distance $d^{\delta_\varepsilon^x u}$, like this:

$$\mid\frac{1}{\varepsilon}d(\delta_\varepsilon^x v,\delta_\varepsilon^x w)\ -\ d^{\delta_\varepsilon^x u}\left(\delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u}\delta_\varepsilon^x v,\delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u}\delta_\varepsilon^x w\right)\mid\to 0 \tag{18}$$

as $\varepsilon\to 0$. By the axioms A1, A3, the function

$$(x,u,v)\mapsto d^x(u,v)$$

is an uniform limit of continuous functions, therefore uniformly continuous on compact sets. We can pass to the limit in the left hand side of the estimate (18), using this uniform continuity and axioms A3, A4, to get the result. □

Corollary 7.5 *If for any x the distance d^x is non degenerate then there exists $C>0$ such that: for any x and u with $d(x,u)\leq C$ there exists a d^x isometry $\Sigma^x(u,\cdot)$ obtained as the limit:*

$$\lim_{\varepsilon\to 0}\Sigma_\varepsilon^x(u,v)=\Sigma^x(u,v)$$

uniformly with respect to x,u,v in compact set.

Proof. From theorem 7.4 we know that $\Delta^x(u,\cdot)$ is a d^x isometry. If d^x is non degenerate then $\Delta^x(u,\cdot)$ is invertible. Let $\Sigma^x(u,\cdot)$ be the inverse.

From proposition 7.3 we know that $\Sigma_\varepsilon^x(u,\cdot)$ is the inverse of $\Delta_\varepsilon^x(u,\cdot)$. Therefore

$$d^x(\Sigma_\varepsilon^x(u,w),\Sigma^x(u,w))=d^x(\Delta^x(u,\Sigma_\varepsilon^x(u,w)),w)=$$

$$=d^x(\Delta^x(u,\Sigma_\varepsilon^x(u,w)),\Delta_\varepsilon^x(u,\Sigma_\varepsilon^x(u,w)).$$

From the uniformity of convergence in theorem 7.4 and the uniformity assumptions in axioms of dilation structures, the conclusion follows. □

The next theorem is the generalization of proposition 3.5.

Theorem 7.6 *Let (X,d,δ) be a dilation structure (which satisfies the strong form of the axiom A2), such that for any $x\in X$ the distance d^x is non degenerate. Then for any $x\in X$ $(U(x),\Sigma^x,\delta^x)$ is a conical group. Moreover, left translations of this group are d^x isometries.*

Proof. We start by proving that $(U(x), \Sigma^x)$ is a local uniform group. The uniformities are induced by the distance d.

We shall use the general relations written in terms of binary decorated trees. Indeed, according to proposition 7.3, we can pass to the limit with $\varepsilon \to 0$ and define:

$$inv^x(u) = \lim_{\varepsilon \to 0} \Delta^x_\varepsilon(u, x) = \Delta^x(u, x).$$

From relation (d) , proposition (7.3) we get (after passing to the limit with $\varepsilon \to 0$)

$$inv^x(inv^x(u)) = u.$$

We shall see that $inv^x(u)$ is the inverse of u. Relation (c), proposition (7.3) gives:

$$\Delta^x(u, v) = \Sigma^x(inv^x(u), v) \tag{19}$$

therefore relations (a), (b) from proposition 7.3 give

$$\Sigma^x(inv^x(u), \Sigma^x(u, v)) = v, \tag{20}$$

$$\Sigma^x(u, \Sigma^x(u, v)) = v. \tag{21}$$

Relation (e) from proposition 7.3 gives

$$\Sigma^x(u, \Sigma^x(v, w)) = \Sigma^x(\Sigma^x(u, v), w) \tag{22}$$

which shows that Σ^x is an associative operation. From (21), (20) we obtain that for any u, v

$$\Sigma^x(\Sigma^x(inv^x(u), u), v) = v, \tag{23}$$

$$\Sigma^x(\Sigma^x(u, inv^x(u)), v) = v. \tag{24}$$

Remark that for any x, v and $\varepsilon \in (0, 1)$ we have $\Sigma^x(x, v) = v$. Therefore x is a neutral element at left for the operation Σ^x. From the definition of inv^x, relation (19) and the fact that inv^x is equal to its inverse, we get that x is an inverse at right too: for any x, v we have

$$\Sigma^x(v, x) = v.$$

Replace now v by x in relations (23), (24) and prove that indeed $inv^x(u)$ is the inverse of u.

We still have to prove that $(U(x), \Sigma^x)$ admits δ^x as dilations.In this reasoning we need the axiom A2 in strong form.

Namely we have to prove that for any $\mu \in (0, 1)$ we have

$$\delta^x_\mu \Sigma^x(u, v) = \Sigma^x(\delta^x_\mu u, \delta^x_\mu v).$$

For this is sufficient to notice that

$$\delta^x_\mu \Delta^x_{\varepsilon\mu}(u, v) = \Delta^x_\varepsilon(\delta^x_\mu u, \delta^x_\mu v)$$

and pass to the limit as $\varepsilon \to 0$. Notice that here we used the fact that dilations δ^x_ε and δ^x_μ exactly commute (axiom A2 in strong form).

Finally, left translations L^x_u are d^x isometries. Indeed, this is a straightforward consequence of theorem 7.4 and corollary 7.5. □

Definition 7.7 *The (local) conical group $(U(x), \Sigma^x, \delta^x)$ can be seen as the tangent space of (X, d, δ) at x. We shall denote it by $T_x(X, d, \delta) = (U(x), \Sigma^x, \delta^x)$, or by T_xX if (d, δ) are clear from the context.*

7.3 Topological considerations

In this subsection we compare various topologies and uniformities related to a dilation structure.

The axiom A3 implies that for any $x \in X$ the function d^x is continuous, therefore open sets with respect to d^x are open with respect to d.

If (X, d) is separable and d^x is non degenerate then $(U(x), d^x)$ is also separable and the topologies of d and d^x are the same. Therefore $(U(x), d^x)$ is also locally compact (and a set is compact with respect to d^x if and only if it is compact with respect to d).

If (X, d) is separable and d^x is non degenerate then the uniformities induced by d and d^x are the same. Indeed, let $\{u_n : n \in \mathbb{N}\}$ be a dense set in $U(x)$, with $x_0 = x$. We can embed $(U(x), (\delta^x, \varepsilon))$ isometrically in the separable Banach space l^∞, for any $\varepsilon \in (0, 1)$, by the function

$$\phi_\varepsilon(u) = \left(\frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x x_n) - \frac{1}{\varepsilon} d(\delta_\varepsilon^x x, \delta_\varepsilon^x x_n) \right)_n .$$

A reformulation of point (a) in theorem 7.1 is that on compact sets ϕ_ε uniformly converges to the isometric embedding of $(U(x), d^x)$

$$\phi(u) = \left(d^x(u, x_n) - d^x(x, x_n) \right)_n .$$

Remark that the uniformity induced by (δ, ε) is the same as the uniformity induced by d, and that it is the same induced from the uniformity on l^∞ by the embedding ϕ_ε. We proved that the uniformities induced by d and d^x are the same.

From previous considerations we deduce the following characterization of tangent spaces associated to a dilation structure.

Corollary 7.8 *Let (X, d, δ) be a strong dilation structure with group $\Gamma = (0, +\infty)$. Then for any $x \in X$ the local group $(U(x), \Sigma^x)$ is locally a simply connected Lie group whose Lie algebra admits a positive graduation (a homogeneous group).*

Proof. Use the facts: $(U(x), \Sigma^x)$ is a locally compact group (from previous topological considerations) which admits δ^x as a contractive automorphism group (from theorem 7.6). Apply then Siebert proposition 3.11 (which is [29] proposition 5.4). □

Remark 7.9 *S. Vodopyanov (private communication) made the observation that in the proof of corollary 6.3 [6] it is used Siebert' proposition 5.4 [29], which is true for conical groups (in our language), while I am using it for* local *conical groups. This is true and constitutes a gap in the proof of the corollary 6.3. Fortunately the recent paper [18] provides the needed result for local groups. Indeed, theorem 1.1 [18] states that a locally compact, locally connected, contractible (with Siebert' wording) group is locally isomorphic to a contractive Lie group.*

7.4 Tangent bundle of a dilation structure

The following two theorems describe the most important metric and algebraic properties of a dilation structure. As presented here these are condensed statements, available in full length as theorems 7, 8, 10 in [5]. The first theorem does not need a proof (see theorem 7 [5]).

Straightforward modifications in the proof of the before mentioned theorems allow us to extend some results to length dilation structures.

Theorem 7.10 *Let (X, d, δ) be a strong dilation structure or a length dilation structure. Then:*

(a) Σ^x *is a local group operation on* $U(x)$*, with* x *as neutral element and* inv^x *as the inverse element function;*

(b) for strong dilation structures the distance d^x *is left invariant with respect to the group operation from point (a); for length dilation structures the length functional* $l^x = l(x, \cdot)$ *is invariant with respect to left translations* $\Sigma^x(y, \cdot)$*,* $y \in U(x)$*;*

(c) For any $\varepsilon \in (0, 1]$ *the dilation* δ^x_ε *is an automorphism with respect to the group operation from point (a);*

(d) for strong dilation structures the distance d^x *has the cone property with respect to dilations: for any* $u, v \in X$ *such that* $d(x, u) \leq 1$ *and* $d(x, v) \leq 1$ *and all* $\mu \in (0, A)$ *we have:*

$$d^x(u, v) \ = \ \frac{1}{\mu} d^x(\delta^x_\mu u, \delta^x_\mu v)$$

For length dilation structures we have for any $\mu \in (0, 1]$ *the equality*

$$l(x, \delta^x_\mu c) \ = \ \mu \, l(x, c)$$

Proof. We shall only prove the statements concerning length dilation structures. For (a) and (c) notice that the axiom A4+ is all that we need in order to transform the proof of theorem 10 [5] into a proof of this point. Indeed, for this we need the existence of the limits from A4+ and the algebraic relations from theorem 11 [5] which are true only from A0, A1, A2.

For (b) remark that if $(\delta^x_\varepsilon y, c) \in \mathcal{L}_\varepsilon(X, d, \delta)$ then $(x, \Sigma^x_\varepsilon(y, \cdot)c) \in \mathcal{L}_\varepsilon(X, d, \delta)$ and moreover

$$l_\varepsilon(\delta^x_\varepsilon y, c) \ = \ l_\varepsilon(x, \Sigma^x_\varepsilon(y, \cdot)c)$$

Indeed, this is true because of the equality:

$$\delta^{\delta^x_\varepsilon y} c \ = \ \delta^x_\varepsilon \Sigma^x_\varepsilon(y, \cdot)c$$

By passing to the limit with $\varepsilon \to 0$ and using A3L and A4+ we get

$$l(x, c) \ = \ l(x, \Sigma^x(y, \cdot)c)$$

For (d) remark that for any $\varepsilon, \mu > 0$ (and sufficiently small) $(x, c) \in \mathcal{L}_{\varepsilon\mu}(X, d, \delta)$ is equivalent with $(x, \delta^x_\mu c) \in \mathcal{L}_\varepsilon(X, d, \delta)$ and moreover:

$$l_\varepsilon(x, \delta^x_\mu c) \ = \ \frac{1}{\varepsilon} \, l_d(\delta^x_{\varepsilon\mu} c) \ = \ \mu \, l_{\varepsilon\mu}(x, c)$$

We pass to the limit with $\varepsilon \to 0$ and we get the desired equality. □

7.5 Differentiability with respect to dilation structures

For any strong dilation structure or length dilation structure there is an associated notion of differentiability (section 7.2 [5]). First we need the definition of a morphism of conical groups.

Definition 7.11 *Let* (N, δ) *and* $(M, \bar{\delta})$ *be two conical groups. A function* $f : N \to M$ *is a conical group morphism if* f *is a group morphism and for any* $\varepsilon > 0$ *and* $u \in N$ *we have* $f(\delta_\varepsilon u) = \bar{\delta}_\varepsilon f(u)$*.*

The definition of the derivative, or differential, with respect to dilations structures follows. In the case of a pair of Carnot groups this is just the definition of the Pansu derivative introduced in [28].

Definition 7.12 *Let (X, d, δ) and $(Y, \overline{d}, \overline{\delta})$ be two strong dilation structures or length and $f : X \to Y$ be a continuous function. The function f is differentiable in x if there exists a conical group morphism $D f(x) : T_x X \to T_{f(x)} Y$, defined on a neighbourhood of x with values in a neighbourhood of $f(x)$ such that*

$$\lim_{\varepsilon \to 0} \sup \left\{ \frac{1}{\varepsilon} \overline{d} \left(f\left(\delta_\varepsilon^x u\right), \overline{\delta}_\varepsilon^{f(x)} D f(x)(u) \right) \ : \ d(x,u) \leq \varepsilon \right\} = 0, \tag{25}$$

The morphism $D f(x)$ is called the derivative, or differential, of f at x.

The definition also makes sense if the function f is defined on a open subset of (X, d).

8 Dilation structures on sub-riemannian manifolds

In [7] we proved that we can associate dilation structures to regular sub-Riemannian manifolds. This result, explained further, is the source of inspiration of the notion of a coherent projection (section 11).

Here we show that normal frames are the central objects in the establishment of fundamental properties in sub-riemannian geometry, in the following precise sense. We prove that for regular sub-riemannian manifolds, the existence of normal frames (definition 8.7) implies that induced dilation structures exist (theorems 8.9, 8.10). The existence of normal frames has been proved by Bellaïche [3], starting with theorem 4.15 and ending in the first half of section 7.3 (page 62). From these facts all classical results concerning the structure of the tangent space to a point of a regular sub-riemannian manifold can be deduced as straightforward consequences of the theorems which describe the algebraic structure of the tangent space at a point as being a (local) normed conical group.

The purpose is twofold: (a) we try to show that basic results in sub-riemannian geometry are particular cases of the abstract theory of dilation structures, and (b) we try to minimize the contribution of classical differential calculus in the proof of these basic results, by showing that in fact the differential calculus on the sub-riemannian manifold is needed only for proving that normal frames exist and after this stage an intrinsic way of reasoning is possible.

If we take the point of view of Gromov, that the only intrinsic object on a sub-riemannian manifold is the Carnot-Carathéodory distance, the underlying differential structure of the manifold is clearly not intrinsic. Nevertheless in all proofs that I know this differential structure is heavily used. Here we try to prove that in fact it is sufficient to take as intrinsic objects of sub-riemannian geometry the Carnot-Carathéodory distance and dilation structures compatible with it.

The closest results along these lines are maybe the ones of Vodopyanov. There is a clear correspondence between his way of defining the tangent bundle of a sub-riemannian manifold and the way of dilation structures. In both cases the tangent space to a point is defined only locally, as a neighbourhood of the point, in the manifold, endowed with a local group operation. Vodopyanov proves the existence of the (locally defined) operation under very weak regularity assumptions on the sub-riemannian manifold. The main tool of his proofs is nevertheless the differential structure of the underlying manifold. In distinction, we prove in [5], in an abstract setting, that the very existence of a dilation structure induces a locally defined operation. Here we show that the differential structure of the underlying manifold is important only in order to prove that dilation structures can indeed be constructed from normal frames.

Let M be a connected n dimensional real manifold. A distribution is a smooth subbundle D of M. To any point $x \in M$ there is associated the vector space $D_x \subset T_x M$. The dimension of the

distribution D at point $x \in M$ is denoted by

$$m(x) = dim\, D_x$$

The distribution is smooth, therefore the function $x \in M \mapsto m(x)$ is locally constant. We suppose further that the dimension of the distribution is globally constant and we denote it by m (thus $m = m(x)$ for any $x \in M$). Clearly $m \leq n$; we are interested in the case $m < n$.

A horizontal curve $c : [a, b] \to M$ is a curve which is almost everywhere derivable and for almost any $t \in [a, b]$ we have $\dot{c}(t) \in D_{c(t)}$. The class of horizontal curves will be denoted by $Hor(M, D)$.

Further we shall use the following notion of non-integrability of the distribution D.

Definition 8.1 *The distribution D is completely non-integrable if M is locally connected by horizontal curves curves $c \in Hor(M, D)$.*

A sufficient condition for the distribution D to be completely non-integrable is given by Chow condition (C) [16].

Theorem 8.2 *(Chow) Let D be a distribution of dimension m in the manifold M. Suppose there is a positive integer number k (called the rank of the distribution D) such that for any $x \in X$ there is a topological open ball $U(x) \subset M$ with $x \in U(x)$ such that there are smooth vector fields $X_1, ..., X_m$ in $U(x)$ with the property:*

(C) the vector fields $X_1, ..., X_m$ span D_x and these vector fields together with their iterated brackets of order at most k span the tangent space T_yM at every point $y \in U(x)$.

Then the distribution D is completely non-integrable in the sense of definition 8.1.

Definition 8.3 *A sub-riemannian (SR) manifold is a triple (M, D, g), where M is a connected manifold, D is a completely non-integrable distribution on M, and g is a metric (Euclidean inner-product) on the distribution (or horizontal bundle) D.*

8.1 The Carnot-Carathéodory distance

Given a distribution D which satisfies the hypothesis of Chow theorem 8.2, let us consider a point $x \in M$, its neighbourhood $U(x)$, and the vector fields $X_1, ..., X_m$ satisfying the condition (C).

One can define on $U(x)$ a filtration of bundles as follows. Define first the class of horizontal vector fields on U:

$$\mathcal{X}^1(U(x), D) \;=\; \{X \in \mathcal{X}^\infty(U) : \forall y \in U(x)\ ,\ X(y) \in D_y\}$$

Next, define inductively for all positive integers j:

$$\mathcal{X}^{j+1}(U(x), D) \;=\; \mathcal{X}^j(U(x), D) \,+\, [\mathcal{X}^1(U(x), D), \mathcal{X}^j(U(x), D)]$$

Here $[\cdot, \cdot]$ denotes the bracket of vector fields. We obtain therefore a filtration $\mathcal{X}^j(U(x), D) \subset \mathcal{X}^{j+1}(U(x), D)$. Evaluate now this filtration at $y \in U(x)$:

$$V^j(y, U(x), D) \;=\; \left\{X(y) :\, X \in \mathcal{X}^j(U(x), D)\right\}$$

According to Chow theorem there is a positive integer k such that for all $y \in U(x)$ we have

$$D_y = V^1(y, U(x), D) \subset V^2(y, U(x), D) \subset ... \subset V^k(y, U(x), D) = T_yM$$

Consequently, to the sub-riemannian manifold is associated the string of numbers:

$$\nu_1(y) = \dim V^1(y, U(x), D) < \nu_2(y) = \dim V^2(y, U(x), D) < ... < n = \dim M$$

Generally k, $\nu_j(y)$ may vary from a point to another.

The number k is called the step of the distribution at y.

Definition 8.4 *The distribution D is regular if $\nu_j(y)$ are constant on the manifold M. The sub-riemannian manifold $M, D, g)$ is regular if D is regular and for any $x \in M$ there is a topological ball $U(x) \subset M$ with $x \in U(M)$ and an orthonormal (with respect to the metric g) family of smooth vector fields $\{X_1, ..., X_m\}$ in $U(x)$ which satisfy the condition (C).*

The lenght of a horizontal curve is

$$l(c) \ = \ \int_a^b \left(g_{c(t)}(\dot{c}(t), \dot{c}(t))\right)^{\frac{1}{2}} \ \mathrm{d}t$$

The length depends on the metric g.

Definition 8.5 *The Carnot-Carathéodory distance (or CC distance) associated to the sub-riemannian manifold is the distance induced by the length l of horizontal curves:*

$$d(x, y) \ = \ \inf \{l(c) \ : \ c \in Hor(M, D) \ , \ c(a) = x \ , \ c(b) = y\}$$

The Chow theorem ensures the existence of a horizontal path linking any two sufficiently closed points, therefore the CC distance is locally finite. The distance depends only on the distribution D and metric g, and not on the choice of vector fields $X_1, ..., X_m$ satisfying the condition (C). The space (M, d) is locally compact and complete, and the topology induced by the distance d is the same as the topology of the manifold M. (These important details may be recovered from reading carefully the constructive proofs of Chow theorem given by Bellaïche [3] or Gromov [22].)

8.2 Normal frames

In the following we stay in a small open neighbourhood of an arbitrary, but fixed point $x_0 \in M$. All results are local in nature (that is they hold for some small open neighbourhood of an arbitrary, but fixed point of the manifold M). That is why we shall no longer mention the dependence of various objects on x_0, on the neighbourhood $U(x_0)$, or the distribution D.

We shall work further only with regular sub-riemannian manifolds, if not otherwise stated. The topological dimension of M is denoted by n, the step of the regular sub-riemannian manifold (M, D, g) is denoted by k, the dimension of the distribution is m, and there are numbers ν_j, $j = 1, ..., k$ such that for any $x \in M$ we have $dim\, V^j(x) = \nu_j$. The Carnot-Carathéodory distance is denoted by d.

Definition 8.6 *An adapted frame $\{X_1, ..., X_n\}$ is a collection of smooth vector fields which is obtained by the construction described below.*

We start with a collection $X_1, ..., X_m$ of vector fields which satisfy the condition (C). In particular for any point x the vectors $X_1(x), ..., X_m(x)$ form a basis for D_x. We further associate to any word $a_1....a_q$ with letters in the alphabet $1, ..., m$ the multi-bracket $[X_{a_1}, [..., X_{a_q}]...]$.

One can add, in the lexicographic order, $n - m$ elements to the set $\{X_1, ..., X_m\}$ until we get a collection $\{X_1, ..., X_n\}$ such that: for any $j = 1, ..., k$ and for any point x the set $\left\{X_1(x), ..., X_{\nu_j}(x)\right\}$ is a basis for $V^j(x)$.

Let $\{X_1, ..., X_n\}$ be an adapted frame. For any $j = 1, ..., n$ the degree $deg\, X_j$ of the vector field X_j is defined as the only positive integer p such that for any point x we have

$$X_j(x) \in V_x^p \setminus V^{p-1}(x)$$

Further we define normal frames. The name has been used by Vodopyanov [31], but for a slightly different object. The existence of normal frames in the sense of the following definition is the hardest

technical problem in the classical establishment of sub-riemannian geometry. For the informed reader the referee pointed out that condition (a) Definition 8.7 is a part of the conclusion of Gromov approximation theorem, namely when one point coincides with the center of nilpotentization; also condition (b) is equivalent with a statement of Gromov concerning the convergence of rescaled vector fields to their nilpotentization (an informed reader must at least follow in all details the papers Bellaïche [3] and Gromov [22], where differential calculus in the classical sense is heavily used). Therefore the conditions of Definition 8.7 concentrate that part of the foundations of sub-riemannian geometry which makes use of classical differential calculus.

The key details in the Definition below are uniform convergence assumptions. This is in line with Gromov suggestions in the last section of Bellaïche [3].

Definition 8.7 *An adapted frame $\{X_1, ..., X_n\}$ is a normal frame if the following two conditions are satisfied:*

(a) we have the limit

$$\lim_{\varepsilon \to 0_+} \frac{1}{\varepsilon} d\left(\exp\left(\sum_1^n \varepsilon^{deg\, X_i} a_i X_i \right)(y), y \right) \; = \; A(y,a) \in (0, +\infty)$$

uniformly with respect to y in compact sets and $a = (a_1, ..., a_n) \in W$, with $W \subset \mathbb{R}^n$ compact neighbourhood of $0 \in \mathbb{R}^n$,

(b) for any compact set $K \subset M$ with diameter (with respect to the distance d) sufficiently small, and for any $i = 1, ..., n$ there are functions

$$P_i(\cdot, \cdot, \cdot) : U_K \times U_K \times K \to \mathbb{R}$$

with $U_K \subset \mathbb{R}^n$ a sufficiently small compact neighbourhood of $0 \in \mathbb{R}^n$ such that for any $x \in K$ and any $a, b \in U_K$ we have

$$\exp\left(\sum_1^n a_i X_i \right)(x) \; = \; \exp\left(\sum_1^n P_i(a,b,y) X_i \right) \circ \exp\left(\sum_1^n b_i X_i \right)(x)$$

and such that the following limit exists

$$\lim_{\varepsilon \to 0_+} \varepsilon^{-deg\, X_i} P_i(\varepsilon^{deg\, X_j} a_j, \varepsilon^{deg\, X_k} b_k, x) \in \mathbb{R}$$

and it is uniform with respect to $x \in K$ and $a, b \in U_K$.

The existence of normal frames is proven in Bellaïche [3], starting with theorem 4.15 and ending in the first half of section 7.3 (page 62).

In order to understand normal frames let us look to the case of a Lie group G endowed with a left invariant distribution. The distribution is completely non-integrable if it is generated by the left translation of a vector subspace D of the algebra $\mathfrak{g} = T_e G$ which bracket generates the whole algebra $\mathfrak{g}$. Take $\{X_1, ..., X_m\}$ a collection of $m = dim\, D$ left invariant independent vector fields and define with their help an adapted frame, as explained in definition 8.6. Then the adapted frame $\{X_1, ..., X_n\}$ is in fact normal.

8.3 Sub-riemannian dilation structures

To any normal frame of a regular sub-riemannian manifold we associate a dilation structure. (Technically this is a dilation structure defined only locally, as in the case of riemannian manifolds.)

Definition 8.8 *To any normal frame $\{X_1, ..., X_n\}$ of a regular sub-riemannian manifold (M, D, g) we associate the dilation structure (M, d, δ) defined by: d is the Carnot-Carathéodory distance, and for any point $x \in M$ and any $\varepsilon \in (0, +\infty)$ (sufficiently small if necessary), the dilation δ_ε^x is given by:*

$$\delta_\varepsilon^x \left(\exp \left(\sum_{i=1}^n a_i X_i \right) (x) \right) \ = \ \exp \left(\sum_{i=1}^n a_i \varepsilon^{deg X_i} X_i \right) (x)$$

We shall prove that (M, d, δ) is indeed a dilation structure. This allows us to get the main results concerning the infinitesimal geometry of a regular sub-riemannian manifold.

We only have to prove axioms A3 and A4 of dilation structures. We do this in the next two theorems.

The first theorem is a result similar to Gromov local approximation theorem [22], p. 135, or to Bellaïche theorem 7.32 [3]. Note however that here we take as a hypothesis the existence of a normal frame.

Theorem 8.9 *Consider $X_1, ..., X_n$ a normal frame and the associated dilations provided by definition 8.8. Then axiom A3 of dilation structures is satisfied, that is the limit*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d \left(\delta_\varepsilon^x u, \delta_\varepsilon^x v \right) \ = \ d^x(u, v)$$

exists and it uniform with respect to x,u,v sufficiently closed.

Proof. Let $x, u, v \in M$ be sufficiently closed. We write

$$u \ = \ \exp \left(\sum_1^n u_i X_i \right) (x) \qquad , \quad v \ = \ \exp \left(\sum_1^n v_i X_i \right) (x)$$

we compute, using definition 8.8:

$$\frac{1}{\varepsilon} d \left(\delta_\varepsilon^x u, \delta_\varepsilon^x v \right) \ = \ \frac{1}{\varepsilon} d \left(\delta_\varepsilon^x \exp \left(\sum_1^n u_i X_i \right) (x), \delta_\varepsilon^x \exp \left(\sum_1^n v_i X_i \right) (x) \right) \ =$$

$$= \ \frac{1}{\varepsilon} d \left(\exp \left(\sum_1^n \varepsilon^{deg\, X_i} u_i X_i \right) (x), \exp \left(\sum_1^n \varepsilon^{deg\, X_i} v_i X_i \right) (x) \right) \ = \ A_\varepsilon$$

Let us denote by $u_\varepsilon = \exp \left(\sum_1^n \varepsilon^{deg\, X_i} u_i X_i \right) (x)$. Use the first part of the property (b), definition 8.7 of a normal system, to write further:

$$A_\varepsilon \ = \ \frac{1}{\varepsilon} d \left(u_\varepsilon, \exp \left(\sum_1^n P_i(\varepsilon^{deg\, X_j} v_j, \varepsilon^{deg\, X_k} u_k, x) X_i \right) (u_\varepsilon) \right) \ =$$

$$= \ \frac{1}{\varepsilon} d \left(u_\varepsilon, \exp \left(\sum_1^n \varepsilon^{deg\, X_i} \left(\varepsilon^{-deg\, X_i} \, P_i(\varepsilon^{deg\, X_j} v_j, \varepsilon^{deg\, X_k} u_k, x) \right) X_i \right) (u_\varepsilon) \right)$$

We make a final notation: for any $i = 1, ..., n$

$$a_i^{\varepsilon} \;=\; \varepsilon^{-deg\, X_i}\, P_i(\varepsilon^{deg\, X_j} v_j, \varepsilon^{deg\, X_k} u_k, x)$$

thus we have:

$$\frac{1}{\varepsilon}\, d\left(\delta_{\varepsilon}^{x} u, \delta_{\varepsilon}^{x} v\right) \;=\; \frac{1}{\varepsilon}\, d\left(u_{\varepsilon}, \exp\left(\sum_{1}^{n} \varepsilon^{deg\, X_i} a_i^{\varepsilon} X_i\right)(u_{\varepsilon})\right)$$

By the second part of property (b), definition 8.7, the vector $a^{\varepsilon} \in \mathbb{R}^n$ converges to a finite value $a^0 \in \mathbb{R}^n$, as $\varepsilon \to 0$, uniformly with respect to x, u, v in compact set. In the same time u_{ε} converges to x, as $\varepsilon \to 0$. The proof ends by using property (a), definition 8.7. Indeed, we shall use the key assumption of uniform convergence. With the notations from definition 8.7, for fixed $\eta > 0$ the term

$$B(\eta, \varepsilon) = \frac{1}{\varepsilon}\, d\left(u_{\eta}, \exp\left(\sum_{1}^{n} \varepsilon^{deg\, X_i} a_i^{\eta} X_i\right)(u_{\eta})\right)$$

converges to a real number $A(u_{\eta}, a_{\eta})$ as $\varepsilon \to 0$, uniformly with respect to u_{η} and a_{η}. Since u_{η} converges to x and a_{η} converges to a^0 as $\eta \to 0$, by the uniform convergence assumption in (a), definition 8.7 we get that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\, d\left(\delta_{\varepsilon}^{x} u, \delta_{\varepsilon}^{x} v\right) \;=\; \lim_{\eta \to 0} A(u_{\eta}, a_{\eta}) \;=\; A(x, a^0)$$

The proof is done. $\square$

In the next theorem we prove that axiom A4 of dilation structures is satisfied.

Theorem 8.10 *Consider $X_1, ..., X_n$ a normal frame and the associated dilations provided by definition 8.8. Then axiom A4 of dilation structures is satisfied: as ε tends to 0 the quantity*

$$\Delta_{\varepsilon}^{x}(u, v) \;=\; \delta_{\varepsilon^{-1}}^{\delta_{\varepsilon}^{x} u} \circ \delta_{\varepsilon}^{x}(v)$$

converges, uniformly with respect to x, u, v sufficiently closed.

Proof. We shall use the notations from definition 8.6, 8.7, 8.8.

Let $x, u, v \in M$ be sufficiently closed. We write

$$u \;=\; \exp\left(\sum_{1}^{n} u_i X_i\right)(x) \qquad , \quad v \;=\; \exp\left(\sum_{1}^{n} v_i X_i\right)(x)$$

We compute now $\Delta_{\varepsilon}^{x}(u, v)$:

$$\Delta_{\varepsilon}^{x}(u, v) \;=\; \delta_{\varepsilon^{-1}}^{\exp\left(\sum_{1}^{n} \varepsilon^{deg\, X_i} u_i X_i\right)(x)} \exp\left(\sum_{1}^{n} \varepsilon^{deg\, X_i} v_i X_i\right)(x)$$

Let us denote by $u_{\varepsilon} = \delta_{\varepsilon}^{x} u$. Thus we have

$$\Delta_{\varepsilon}^{x}(u, v) \;=\; \delta_{\varepsilon^{-1}}^{u_{\varepsilon}} \exp\left(\sum_{1}^{n} \varepsilon^{deg\, X_i} v_i X_i\right)(x)$$

We use the first part of the property (b), definition 8.7, in order to write

$$\exp\left(\sum_{1}^{n} \varepsilon^{deg\, X_i} v_i X_i\right)(x) \;=\; \exp\left(\sum_{1}^{n} P_i(\varepsilon^{deg\, X_j} v_j, \varepsilon^{deg\, X_k} u_k, x) X_i\right)(u_{\varepsilon})$$

We finish the computation:

$$\Delta_\varepsilon^x(u,v) \;=\; \exp\left(\sum_1^n \varepsilon^{-\deg X_i} P_i(\varepsilon^{\deg X_j} v_j, \varepsilon^{\deg X_k} u_k, x) X_i\right)(u_\varepsilon)$$

As ε goes to 0 the point u_ε converges to x uniformly with respect to x, u sufficiently closed (as a corollary of the previous theorem, for example). The proof therefore ends by invoking the second part of the property (b), definition 8.7. $\square$

With the help of a normal frame we can therefore prove the existence of strong dilation structures on regular sub-riemannian manifolds.

Theorem 8.11 *Let (M, D, g) be a regular sub-riemannian manifold, $U \subset M$ an open set which admits a normal frame. Define for any $x \in U$ and $\varepsilon > 0$ (sufficiently small if necessary), the dilation δ_ε^x given by:*

$$\delta_\varepsilon^x\left(\exp\left(\sum_{i=1}^n a_i X_i\right)(x)\right) \;=\; \exp\left(\sum_{i=1}^n a_i \varepsilon^{deg X_i} X_i\right)(x)$$

Then (U, d, δ) is a strong dilation structure.

9 The Radon-Nikodym property

Let (X, d, δ) be a strong dilation structure or a length dilation structure. We have then a notion of differentiability for curves in X, seen as continuous functions from (a open interval in) $\mathbb{R}$, with the usual dilation structure, to X with the dilation structure (X, d, δ). Further we want to see what differentiability in the sense of definition 7.12 means for curves. In proposition 9.2 we shall arrive to a kind of intrinsic notion of a distribution in a dilation structure, with the geometrical meaning of a cone of all possible derivatives of curves passing through a point.

Definition 9.1 *In a normed conical group N we shall denote by $D(N)$ the set of all $u \in N$ with the property that $\varepsilon \in ((0,\infty), +) \mapsto \delta_\varepsilon u \in N$ is a morphism of groups.*

$D(N)$ is always non empty, because it contains the neutral element of N. $D(N)$ is also a cone, with dilations δ_ε, and a closed set.

Proposition 9.2 *Let (X, d, δ) be a strong dilation structure or a length dilation structure and let $c : [a, b] \to (X, d)$ be a continuous curve. For any $x \in X$ and any $y \in T_x(X, d, \delta)$ we denote by*

$$inv^x(y) \;=\; \Delta^x(y, x)$$

the inverse of y with respect to the group operation in $T_x(X, d, \delta)$. Then the following are equivalent:

(a) c is derivable in $t \in (a, b)$ with respect to the dilation structure (X, d, δ);

(b) there exists $\dot{c}(t) \in D(T_{c(t}(X, d, \delta))$ such that

$$\frac{1}{\varepsilon} d(c(t+\varepsilon), \delta_\varepsilon^{c(t)} \dot{c}(t)) \to 0$$

$$\frac{1}{\varepsilon} d(c(t-\varepsilon), \delta_\varepsilon^{c(t)} inv^{c(t)}(\dot{c}(t))) \to 0$$

Proof. It is straightforward that a conical group morphism $f : \mathbb{R} \to N$ is defined by its value $f(1) \in N$. Indeed, for any $a > 0$ we have $f(a) = \delta_a f(1)$ and for any $a < 0$ we have $f(a) = \delta_a f(1)^{-1}$. From the morphism property we also deduce that

$$\delta v = \left\{ \delta_a v \,:\, a > 0, v = f(1) \text{ or } v = f(1)^{-1} \right\}$$

is a one parameter group and that for all $\alpha, \beta > 0$ we have $\delta_{\alpha+\beta} u = \delta_\alpha u \, \delta_\beta u$. We have therefore a bijection between conical group morphisms $f : \mathbb{R} \to (N, \delta)$ and elements of $D(N)$.

The curve $c : [a, b] \to (X, d)$ is derivable in $t \in (a, b)$ if and only if there is a morphism of normed conical groups $f : \mathbb{R} \to T_{c(t}(X, d, \delta)$ such that for any $a \in \mathbb{R}$ we have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d(c(t + \varepsilon a), \delta_\varepsilon^{c(t)} f(a)) \;=\; 0$$

Take $\dot{c}(t) = f(1)$. Then $\dot{c}(t) \in D(T_{c(t}(X, d, \delta))$. For any $a > 0$ we have $f(a) = \delta_a^{c(t)} \dot{c}(t)$; otherwise if $a < 0$ we have $f(a) = \delta_a^{c(t)} \, inv^{c(t)} \, \dot{c}(t)$. This implies the equivalence stated on the proposition. □

Definition 9.3 *A strong dilation structure or a length dilation structure (X, d, δ) has the* **Radon-Nikodym property (or rectifiability property, or RNP)** *if any Lipschitz curve $c : [a, b] \to (X, d)$ is derivable almost everywhere.*

9.1 Two examples

The following two easy examples will show that not any strong dilation structure has the Radon-Nikodym property.

For $(X, d) = (\mathbb{V}, d)$, a real, finite dimensional, normed vector space, with distance d induced by the norm, the (usual) dilations δ_ε^x are given by:

$$\delta_\varepsilon^x y \;=\; x + \varepsilon(y - x)$$

Dilations are defined everywhere.

There are few things to check: axioms 0,1,2 are obviously true. For axiom A3, remark that for any $\varepsilon > 0$, $x, u, v \in X$ we have:

$$\frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) \;=\; d(u, v) \;,$$

therefore for any $x \in X$ we have $d^x = d$.

Finally, let us check the axiom A4. For any $\varepsilon > 0$ and $x, u, v \in X$ we have

$$\delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v \;=\; x + \varepsilon(u - x) + \frac{1}{\varepsilon} \left(x + \varepsilon(v - x) - x - \varepsilon(u - x) \right) \;=$$

$$=\; x + \varepsilon(u - x) + v - u$$

therefore this quantity converges to

$$x + v - u \;=\; x + (v - x) - (u - x)$$

as $\varepsilon \to 0$. The axiom A4 is verified.

This dilation structure has the Radon-Nikodym property.

Further is an example of a dilation structure which does not have the Radon-Nikodym property. Take $X = \mathbb{R}^2$ with the euclidean distance d. For any $z \in \mathbb{C}$ of the form $z = 1 + i\theta$ we define dilations

$$\delta_\varepsilon x = \varepsilon^z x \;.$$

It is easy to check that $(\mathbb{R}^2, d, \delta)$ is a dilation structure, with dilations

$$\delta_\varepsilon^x y = x + \delta_\varepsilon(y - x)$$

Two such dilation structures (constructed with the help of complex numbers $1 + i\theta$ and $1 + i\theta'$) are equivalent if and only if $\theta = \theta'$.

There are two other interesting properties of these dilation structures. The first is that if $\theta \neq 0$ then there are no non trivial Lipschitz curves in X which are differentiable almost everywhere. It means that such dilation structure does not have the Radon-Nikodym property.

The second property is that any holomorphic and Lipschitz function from X to X (holomorphic in the usual sense on $X = \mathbb{R}^2 = \mathbb{C}$) is differentiable almost everywhere, but there are Lipschitz functions from X to X which are not differentiable almost everywhere (suffices to take a $\mathcal{C}^\infty$ function from $\mathbb{R}^2$ to $\mathbb{R}^2$ which is not holomorphic).

9.2 Length formula from Radon-Nikodym property

Theorem 9.4 *Let (X, d, δ) be a strong dilation structure with the Radon-Nikodym property, over a complete length metric space (X, d). Then for any $x, y \in X$ we have*

$$d(x,y) \;=\; \inf\left\{ \int_a^b d^{c(t)}(c(t), \dot{c}(t))\ dt \ :\ c : [a,b] \to X \ \textit{Lipschitz}\ ,\right.$$

$$\left. c(a) = x, c(b) = y \right\}$$

Proof. From theorem 6.8 we deduce that for almost every $t \in (a, b)$ the upper dilation of c in t can be expressed as:

$$Lip(c)(t) = \lim_{s \to t} \frac{d(c(s), c(t))}{\mid s - t \mid}$$

If the dilation structure has the Radon-Nikodym property then for almost every $t \in [a, b]$ there is $\dot{c}(t) \in D(T_{c(t)}X)$ such that

$$\frac{1}{\varepsilon} d(c(t + \varepsilon), \delta_\varepsilon^{c(t)} \dot{c}(t)) \to 0$$

Therefore for almost every $t \in [a, b]$ we have

$$Lip(c)(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d(c(t + \varepsilon), c(t)) = d^{c(t)}(c(t), \dot{c}(t))$$

The formula for length follows from here. □

A straightforward consequence is that the distance d is uniquely determined by the "distribution" $x \in X \mapsto D(T_x(X, d, \delta))$ and the function which associates to any $x \in X$ the "norm" $\| \cdot \|_x : D(T_x(X, d, \delta)) \to [0, +\infty)$.

Corollary 9.5 *Let (X, d, δ) and $(X, \bar{d}, \bar{\delta})$ be two strong dilation structures with the Radon-Nikodym property , which are also complete length metric spaces, such that for any $x \in X$ we have $D(T_x(X, d, \delta)) = D(T_x(X, \bar{d}, \bar{\delta}))$ and $d^x(x, u) = \bar{d}^x(x, u)$ for any $u \in D(T_x(X, d, \delta))$. Then $d = \bar{d}$.*

9.3 Equivalent dilation structures and their distributions

Definition 9.6 Two strong dilation structures (X,δ,d) *and* $(X,\overline{\delta},\overline{d})$ **are equivalent** *if*

(a) the identity map $id:(X,d)\to(X,\overline{d})$ *is bilipschitz and*

(b) for any $x\in X$ *there are functions* P^x,Q^x *(defined for* $u\in X$ *sufficiently close to* x*) such that*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\overline{d}\left(\delta_\varepsilon^x u,\overline{\delta}_\varepsilon^x Q^x(u)\right)=0, \tag{26}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}d\left(\overline{\delta}_\varepsilon^x u,\delta_\varepsilon^x P^x(u)\right)=0, \tag{27}$$

uniformly with respect to x*,* u *in compact sets.*

Proposition 9.7 (X,δ,d) *and* $(X,\overline{\delta},\overline{d})$ *are equivalent if and only if*

(a) the identity map $id:(X,d)\to(X,\overline{d})$ *is bilipschitz,*

(b) for any $x\in X$ *there are conical group morphisms:*

$$P^x:T_x(X,\overline{\delta},\overline{d})\to T_x(X,\delta,d)\ \text{and}\ Q^x:T_x(X,\delta,d)\to T_x(X,\overline{\delta},\overline{d})$$

such that the following limits exist

$$\lim_{\varepsilon\to 0}\left(\overline{\delta}_\varepsilon^x\right)^{-1}\delta_\varepsilon^x(u)=Q^x(u), \tag{28}$$

$$\lim_{\varepsilon\to 0}\left(\delta_\varepsilon^x\right)^{-1}\overline{\delta}_\varepsilon^x(u)=P^x(u), \tag{29}$$

and are uniform with respect to x*,* u *in compact sets.*

The next theorem shows a link between the tangent bundles of equivalent dilation structures.

Theorem 9.8 *Let* (X,d,δ) *and* $(X,\overline{d},\overline{\delta})$ *be equivalent strong dilation structures. Then for any* $x\in X$ *and any* $u,v\in X$ *sufficiently close to* x *we have:*

$$\overline{\Sigma}^x(u,v)=Q^x\left(\Sigma^x\left(P^x(u),P^x(v)\right)\right). \tag{30}$$

The two tangent bundles are therefore isomorphic in a natural sense.

As a consequence, the following corollary is straightforward.

Corollary 9.9 *Let* (X,d,δ) *and* $(X,\overline{d},\overline{\delta})$ *be equivalent strong dilation structures. Then for any* $x\in X$ *we have*

$$Q^x(D(T_x(X,\delta,d)))\ =\ D(T_x(X,\overline{\delta},\overline{d}))$$

If (X,d,δ) *has the Radon-Nikodym property , then* $(X,\overline{d},\overline{\delta})$ *has the same property.*

Suppose that (X,d,δ) *and* $(X,\overline{d},\overline{\delta})$ *are complete length spaces with the Radon-Nikodym property . If the functions* P^x,Q^x *from definition 9.6 (b) are isometries, then* $d=\overline{d}$*.*

10 Tempered dilation structures

The notion of a tempered dilation structure is inspired by the results from Venturini [30] and Buttazzo, De Pascale and Fragala [13].

The examples of length dilation structures from this section are provided by the extension of some results from [13] (propositions 2.3, 2.6 and a part of theorem 3.1) to dilation structures.

The following definition gives a class of distances $\mathcal{D}(\Omega, \bar{d}, \bar{\delta})$, associated to a strong dilation structure $(\Omega, \bar{d}, \bar{\delta})$, which generalizes the class of distances $\mathcal{D}(\Omega)$ from [13], definition 2.1.

Definition 10.1 *For any strong dilation structure $(\Omega, \bar{d}, \bar{\delta})$ we define the class $\mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ of all distance functions d on Ω such that*

(a) d is a length distance,

(b) for any $\varepsilon > 0$ and any x, u, v sufficiently close the are constants $0 < c < C$ such that:

$$c\, \bar{d}^x(u, v) \ \leq \ \frac{1}{\varepsilon}\, d(\bar{\delta}^x_\varepsilon u, \bar{\delta}^x_\varepsilon v) \ \leq \ C\, \bar{d}^x(u, v) \tag{31}$$

The dilation structure $(\Omega, \bar{d}, \bar{\delta})$ is **tempered** *if $\bar{d} \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$.*

On $\mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ we put the topology of uniform convergence (induced by distance $\bar{d}$) on compact subsets of $\Omega \times \Omega$.

To any distance $d \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ we associate the function:

$$\phi_d(x, u) \ = \ \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon}\, d(x, \delta^x_\varepsilon u)$$

defined for any $x, u \in \Omega$ sufficiently close. We have therefore

$$c\, \bar{d}^x(x, u) \ \leq \ \phi_d(x, u) \ \leq \ C\, \bar{d}^x(x, u) \tag{32}$$

Notice that if $d \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ then for any x, u, v sufficiently close we have

$$-\bar{d}(x, u)\, O(\bar{d}(x, u)) \ + \ c\, \bar{d}^x(u, v) \ \leq$$

$$\leq \ d(u, v) \ \leq \ C\, \bar{d}^x(u, v) \ + \ \bar{d}(x, u)\, O(\bar{d}(x, u))$$

If $c : [0, 1] \to \Omega$ is a d-Lipschitz curve and $d \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ then we may decompose it in a finite family of curves $c_1, ..., c_n$ (with n depending on c) such that there are $x_1, ..., x_n \in \Omega$ with c_k is $\bar{d}^{x_k}$-Lipschitz. Indeed, the image of the curve $c([0, 1])$ is compact, therefore we may cover it with a finite number of balls $B(c(t_k), \rho_k, \bar{d}^{c(t_k)})$ and apply (31). If moreover $(\Omega, \bar{d}, \bar{\delta})$ is tempered then it follows that $c : [0, 1] \to \Omega$ d-Lipschitz curve is equivalent with c $\bar{d}$-Lipschitz curve.

By using the same arguments as in the proof of theorem 9.4, we get the following extension of proposition 2.4 [13].

Proposition 10.2 *If $(\Omega, \bar{d}, \bar{\delta})$ is tempered, with the Radon-Nikodym property, and $d \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ then*

$$d(x, y) \ = \ \inf \left\{ \int_a^b \phi_d(c(t), \dot{c}(t)) \ dt \ : \ c : [a, b] \to X \ \bar{d}\text{-Lipschitz} \ , \right.$$

$$\left. c(a) = x, c(b) = y \right\}$$

The next theorem is a generalization of a part of theorem 3.1 [13].

Theorem 10.3 *Let $(\Omega, \bar{d}, \bar{\delta})$ be a strong dilation structure which is tempered, with the Radon-Nikodym property, and $d_n \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ a sequence of distances converging to $d \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$. Denote by L_n, L the length functional induced by the distance d_n, respectively by d. Then L_n Γ-converges to L.*

Proof. This is the generalization of the implication (i) $\Rightarrow$ (iii), theorem 3.1 [13]. The proof (p. 252-253) is almost identical, we only need to replace everywhere expressions like $\mid x - y \mid$ by $\bar{d}(x, y)$ and use proposition 10.2, relations (32) and (31) instead of respectively proposition 2.4 and relations (2.6) and (2.3) [13]. □

Using this result we obtain a large class of examples of length dilation structures.

Corollary 10.4 *If $(\Omega, \bar{d}, \bar{\delta})$ is strong dilation structure which is tempered and it has the Radon-Nikodym property then it is a length dilation structure.*

Proof. Indeed, from the hypothesis we deduce that $\bar{\delta}_\varepsilon^x \bar{d} \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$. For any sequence $\varepsilon_n \to 0$ we thus obtain a sequence of distances $d_n = \bar{\delta}_{\varepsilon_n}^x \bar{d}$ converging to $\bar{d}^x$. We apply now theorem 10.3 and we get the result. □

11 Coherent projections

For a given dilation structure with the Radon-Nikodym property, we shall give a procedure to construct another dilation structure, such that the first one looks down to the the second one.

This will be done with the help of coherent projections.

Definition 11.1 *Let $(X, \bar{d}, \bar{\delta})$ be a strong dilation structure. A* **coherent projection** *of $(X, \bar{d}, \bar{\delta})$ is a function which associates to any $x \in X$ and $\varepsilon \in (0, 1]$ a map $Q_\varepsilon^x : U(x) \to X$ such that:*

(I) $Q_\varepsilon^x : U(x) \to Q_\varepsilon^x(U(x))$ is invertible and the inverse will be denoted by $Q_{\varepsilon^{-1}}^x$; for any $\varepsilon, \mu > 0$ and any $x \in X$ we have

$$Q_\varepsilon^x \, \bar{\delta}_\mu^x \;=\; \bar{\delta}_\mu^x \, Q_\varepsilon^x$$

(II) the limit $\lim_{\varepsilon \to 0} Q_\varepsilon^x u = Q^x u$ is uniform with respect to x, u in compact sets.

(III) for any $\varepsilon, \mu > 0$ and any $x \in X$ we have $Q_\varepsilon^x \, Q_\mu^x = Q_{\varepsilon\mu}^x$. Also $Q_1^x = id$ and $Q_\varepsilon^x x = x$.

(IV) define $\Theta_\varepsilon^x(u, v) = \bar{\delta}_{\varepsilon^{-1}}^x \, Q_{\varepsilon^{-1}}^{\bar{\delta}_\varepsilon^x Q_\varepsilon^x u} \, \bar{\delta}_\varepsilon^x Q_\varepsilon^x v$. Then the limit exists

$$\lim_{\varepsilon \to 0} \Theta_\varepsilon^x(u, v) \;=\; \Theta^x(u, v)$$

and it is uniform with respect to x, u, v in compact sets.

Remark 11.2 *Property (IV) is basically a smoothness condition on the coherent projection Q, relative to the strong dilation structure $(X, \bar{d}, \bar{\delta})$.*

Proposition 11.3 *Let $(X, \bar{d}, \bar{\delta})$ be a strong dilation structure and Q a coherent projection. We define $\delta_\varepsilon^x = \bar{\delta}_\varepsilon^x \, Q_\varepsilon^x$. Then:*

(a) for any $\varepsilon, \mu > 0$ and any $x \in X$ we have $\delta_\varepsilon^x \, \bar{\delta}_\mu^x = \bar{\delta}_\mu^x \, \delta_\varepsilon^x$.

(b) for any $x \in X$ we have $Q^x \, Q^x = Q^x$ (thus Q^x is a projection).

(c) δ satisfies the conditions A1, A2, A4 from definition 4.4.

Proof. (a) this is a consequence of the commutativity condition (I) (second part). Indeed, we have $\delta_\varepsilon^x\,\bar\delta_\mu^x \;=\; \bar\delta_\varepsilon^x\,Q_\varepsilon^x\,\bar\delta_\mu^x \;=\; \bar\delta_\varepsilon^x\,\bar\delta_\mu^x\,Q_\varepsilon^x \;=\; \bar\delta_\mu^x\,\bar\delta_\varepsilon^x\,Q_\varepsilon^x \;=\; \bar\delta_\mu^x\,\delta_\varepsilon^x$.

(b) we pass to the limit $\varepsilon\to 0$ in the equality $Q_{\varepsilon^2}^x \;=\; Q_\varepsilon^x\,Q_\varepsilon^x$ and we get, based on condition (II), that $Q^x\,Q^x \;=\; Q^x$.

(c) Axiom A1 for δ is equivalent with (III). Indeed, the equality $\delta_\varepsilon^x\,\delta_\mu^x \;=\; \delta_{\varepsilon\mu}^x$ is equivalent with: $\bar\delta_{\varepsilon\mu}^x\,Q_{\varepsilon\mu}^x \;=\; \bar\delta_{\varepsilon\mu}^x\,Q_\varepsilon^x\,Q_\mu^x$. This is true because $Q_\varepsilon^x\,Q_\mu^x \;=\; Q_{\varepsilon\mu}^x$. We also have $\delta_1^x \;=\; \bar\delta_1^x Q_1^x \;=\; Q_1^x \;=\; id$. Moreover $\delta_\varepsilon^x x \;=\; \bar\delta_\varepsilon^x\,Q_\varepsilon^x x \;=\; Q_\varepsilon^x\,\bar\delta_\varepsilon^x x \;=\; Q_\varepsilon^x x \;=\; x$. Let us compute now:

$$\Delta_\varepsilon^x(u,v) \;=\; \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u}\,\delta_\varepsilon^x v \;=\; \bar\delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u}\,Q_{\varepsilon^{-1}}^{\delta_\varepsilon^x u}\,\delta_\varepsilon^x v \;=$$

$$=\; \bar\delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u}\,\bar\delta_\varepsilon^x\,\Theta_\varepsilon^x(u,v) \;=\; \bar\Delta_\varepsilon^x(Q_\varepsilon^x u,\Theta_\varepsilon^x(u,v))$$

We can pass to the limit in the last term of this string of equalities and we prove that the axiom A4 is satisfied by δ: there exists the limit

$$\Delta^x(u,v) \;=\; \lim_{\varepsilon\to 0}\Delta_\varepsilon^x(u,v) \tag{33}$$

which is uniform as written in A4, moreover we have the equality

$$\Theta_\varepsilon^x(u,v) \;=\; \bar\Sigma_\varepsilon^x(Q_\varepsilon^x u,\Delta_\varepsilon^x(u,v)) \tag{34}$$

□

We collect two useful relations in the next proposition.

Proposition 11.4 *Let $(X,\bar d,\bar\delta)$ be a strong dilation structure and Q a coherent projection. We denote by δ the field of dilations induced by the coherent projection, as in the previous proposition, and by Δ^x is defined by (33). Then we have:*

$$\Delta^x(u,v) \;=\; \bar\Delta^x(Q^x u,\Theta^x(u,v)) \tag{35}$$

$$Q^x\Delta^x(u,v) \;=\; \bar\Delta^x(Q^x u,Q^x v) \tag{36}$$

Proof. After passing to the limit with $\varepsilon\to 0$ in the relation (34) we get the formula (35). In order to prove (36) we notice that:

$$Q_{\varepsilon^{\delta_\varepsilon^x u}}^{\delta_\varepsilon^x u}\,\Delta_\varepsilon^x(u,v) \;=\; Q_{\varepsilon^{\delta_\varepsilon^x u}}^{\delta_\varepsilon^x u}\delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u}\delta_\varepsilon^x v \;=$$

$$=\; \bar\delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u}\bar\delta_\varepsilon^x Q_\varepsilon^x v \;=\; \bar\Delta_\varepsilon^x(Q_\varepsilon^x u,Q_\varepsilon^x v)$$

which gives(36) as we pass to the limit with $\varepsilon\to 0$ in this relation. □

Next is described the notion of Q-horizontal curve.

Definition 11.5 *Let $(X,\bar d,\bar\delta)$ be a strong dilation structure and Q a coherent projection. A curve $c:[a,b]\to X$ is Q-**horizontal** if for almost any $t\in[a,b]$ the curve c is derivable and the derivative of c at t, denoted by $\dot c(t)$ has the property:*

$$Q^{c(t)}\dot c(t) \;=\; \dot c(t) \tag{37}$$

*A curve $c:[a,b]\to X$ is Q-**everywhere horizontal** if for all $t\in[a,b]$ the curve c is derivable and the derivative has the horizontality property (37).*

We shall now use the notations from section 6. We look first at some induced dilation structures.

For any $x \in X$ and $\varepsilon \in (0,1)$ the dilation δ_ε^x can be seen as an isomorphism of strong dilation structures with coherent projections:

$$\delta_\varepsilon^x : (U(x), \delta_\varepsilon^x \bar{d}, \hat{\delta}_\varepsilon^x, \hat{Q}_\varepsilon^x) \to (\delta_\varepsilon^x U(x), \frac{1}{\varepsilon}\bar{d}, \bar{\delta}, Q)$$

which defines the dilations $\hat{\delta}_{\varepsilon,\cdot}^{x,\cdot}$ and coherent projection $\hat{Q}_\varepsilon^x$ by:

$$\begin{aligned} \hat{\delta}_{\varepsilon,\mu}^{x,u} &= \delta_{\varepsilon^{-1}}^x \, \bar{\delta}_\mu^{\delta_\varepsilon^x u} \, \delta_\varepsilon^x \\ \hat{Q}_{\varepsilon,\mu}^{x,u} &= \delta_{\varepsilon^{-1}}^x \, Q_\mu^{\delta_\varepsilon^x u} \, \delta_\varepsilon^x \end{aligned}$$

Also the dilation $\bar{\delta}_\varepsilon^x$ is an isomorphism of strong dilation structures with coherent projections:

$$\bar{\delta}_\varepsilon^x : (U(x), \bar{\delta}_\varepsilon^x \bar{d}, \bar{\delta}_\varepsilon^x, \bar{Q}_\varepsilon^x) \to (\bar{\delta}_\varepsilon^x U(x), \frac{1}{\varepsilon}\bar{d}, \bar{\delta}, Q)$$

which defines the dilations $\bar{\delta}_{\varepsilon,\cdot}^{x,\cdot}$ and coherent projection $\bar{Q}_\varepsilon^x$ by:

$$\begin{aligned} \bar{\delta}_{\varepsilon,\mu}^{x,u} &= \bar{\delta}_{\varepsilon^{-1}}^x \, \bar{\delta}_\mu^{\bar{\delta}_\varepsilon^x u} \, \bar{\delta}_\varepsilon^x \\ \bar{Q}_{\varepsilon,\mu}^{x,u} &= \bar{\delta}_{\varepsilon^{-1}}^x \, Q_\mu^{\bar{\delta}_\varepsilon^x u} \, \bar{\delta}_\varepsilon^x \end{aligned}$$

Because $\delta_\varepsilon^x = \bar{\delta}_\varepsilon^x \, Q_\varepsilon^x$ we get that

$$Q_\varepsilon^x : (U(x), \delta_\varepsilon^x \bar{d}, \hat{\delta}_\varepsilon^x, \hat{Q}_\varepsilon^x) \to (Q_\varepsilon^x U(x), \bar{\delta}_\varepsilon^x d, \bar{\delta}_\varepsilon^x, \bar{Q}_\varepsilon^x)$$

is an isomorphism of strong dilation structures with coherent projections.

Further is a useful description of the coherent projection $\hat{Q}_\varepsilon^x$.

Proposition 11.6 *With the notations previously made, for any $\varepsilon \in (0,1]$, $x, u, v \in X$ sufficiently close and $\mu > 0$ we have:*

(i) $\hat{Q}_{\varepsilon,\mu}^{x,u} v = \Sigma_\varepsilon^x (u, Q_\mu^{\delta_\varepsilon^x u} \Delta_\varepsilon^x (u,v))$,

(ii) $\hat{Q}_\varepsilon^{x,u} v = \Sigma_\varepsilon^x (u, Q^{\delta_\varepsilon^x u} \Delta_\varepsilon^x (u,v))$.

Proof. (i) implies (ii) when $\mu \to 0$, thus it is sufficient to prove only the first point. This is the result of a computation:

$$\begin{aligned} \hat{Q}_{\varepsilon,\mu}^{x,u} v &= \delta_{\varepsilon^{-1}}^x \, Q_\mu^{\delta_\varepsilon^x u} \, \delta_\varepsilon^x = \\ &= \delta_{\varepsilon^{-1}}^x \, \delta_\varepsilon^{\delta_\varepsilon^x u} \, Q_\mu^{\delta_\varepsilon^x u} \, \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \, \delta_\varepsilon^x = \Sigma_\varepsilon^x (u, Q_\mu^{\delta_\varepsilon^x u} \Delta_\varepsilon^x (u,v)) \end{aligned}$$

□

Notation concerning derivatives. We shall denote the derivative of a curve with respect to the dilations $\hat{\delta}_\varepsilon^x$ by $\frac{\hat{d}_\varepsilon^x}{dt}$. Also, the derivative of the curve c with respect to $\bar{\delta}$ is denoted by $\frac{\bar{d}}{dt}$, and so on.

By computation we get: the curve c is $\hat{\delta}_\varepsilon^x$-derivable if and only if $\delta_\varepsilon^x c$ is $\bar{\delta}$-derivable and

$$\frac{\hat{d}_\varepsilon^x}{dt} c(t) = \delta_{\varepsilon^{-1}}^x \frac{\bar{d}}{dt} (\delta_\varepsilon^x c)(t)$$

With these notations we give a proposition which explains that the operator Θ_ε^x , from the definition of coherent projections, is a lifting operator.

Proposition 11.7 *If the curve $\delta_\varepsilon^x c$ is Q-horizontal then*

$$\frac{\bar{d}_\varepsilon^x}{dt}\left(Q_\varepsilon^x c\right)(t) \;=\; \Theta_\varepsilon^x(c(t), \frac{\hat{d}_\varepsilon^x}{dt}c(t))$$

Proof. If the curve $Q_\varepsilon^x c$ is $\bar{\delta}_\varepsilon^x$ derivable and $\bar{Q}_\varepsilon^x$ horizontal. We have therefore:

$$\frac{\bar{d}_\varepsilon^x}{dt}\left(Q_\varepsilon^x c\right)(t) \;=\; \bar{\delta}_{\varepsilon^{-1}}^x\, Q^{\delta_\varepsilon^x c(t)}\, \bar{\delta}_\varepsilon^x\, \frac{\bar{d}_\varepsilon^x}{dt}\left(Q_\varepsilon^x c\right)(t)$$

which implies:

$$\bar{\delta}_\varepsilon^x\, \frac{\bar{d}_\varepsilon^x}{dt}\left(Q_\varepsilon^x c\right)(t) \;=\; Q_{\varepsilon^{-1}}^{\delta_\varepsilon^x c(t)}\, \bar{\delta}_\varepsilon^x\, \frac{\bar{d}_\varepsilon^x}{dt}\left(Q_\varepsilon^x c\right)(t) \;=\; Q_{\varepsilon^{-1}}^{\delta_\varepsilon^x c(t)}\, \delta_\varepsilon^x\, \frac{\hat{d}_\varepsilon^x}{dt}c(t)$$

which is the formula we wanted to prove. □

11.1 Distributions in sub-riemannian spaces

The inspiration for the notion of coherent projection comes from sub-riemannian geometry. We shall look to the section 8 with a fresh eye.

Further we shall work locally, just as in the mentioned section. Same notations are used. Let $\{Y_1, ..., Y_n\}$ be a frame induced by a parameterization $\phi : O \subset \mathbb{R}^n \to U \subset M$ of a small open, connected set U in the manifold M. This parameterization induces a affine dilation structure on U, by

$$\tilde{\delta}_\varepsilon^{\phi(a)}\, \phi(b) \;=\; \phi\left(a + \varepsilon(-a + b)\right)$$

We take the distance $\tilde{d}(\phi(a), \phi(b)) \;=\; \|b - a\|$.

Let $\{X_1, ..., X_n\}$ be a normal frame, cf. definition 8.7, d be the Carnot-Carathéodory distance and

$$\delta_\varepsilon^x\left(\exp\left(\sum_{i=1}^n a_i X_i\right)(x)\right) \;=\; \exp\left(\sum_{i=1}^n a_i \varepsilon^{degX_i} X_i\right)(x)$$

be the dilation structure associated, by theorem 8.11.

We may take another dilation structure, constructed as follows: extend the metric g on the distribution D to a riemannian metric on M, denoted for convenience also by g. Let $\bar{d}$ be the riemannian distance induced by the riemannian metric g, and the dilations

$$\bar{\delta}_\varepsilon^x\left(\exp\left(\sum_{i=1}^n a_i X_i\right)(x)\right) \;=\; \exp\left(\sum_{i=1}^n a_i \varepsilon X_i\right)(x)$$

Then $(U, \bar{d}, \bar{\delta})$ is a strong dilation structure which is equivalent with the dilation structure $(U, \tilde{d}, \tilde{\delta})$.

From now we may define coherent projections associated either to the pair $(\tilde{\delta}, \delta)$ or to the pair $(\bar{\delta}, \delta)$. Because we put everything on the manifold (by the use of the chosen frames), we shall obtain different coherent projections, both inducing the same dilation structure (U, d, δ).

Let us define Q_ε^x by:

$$Q_\varepsilon^x\left(\exp\left(\sum_{i=1}^n a_i X_i\right)(x)\right) \;=\; \exp\left(\sum_{i=1}^n a_i \varepsilon^{degX_i - 1} X_i\right)(x) \tag{38}$$

Proposition 11.8 *Q is a coherent projection associated with the dilation structure $(U, \bar{d}, \bar{\delta})$.*

Proof. (I) definition 11.1 is true, because $\delta_\varepsilon^x\, u = Q_\varepsilon^x\, \bar{\delta}_\varepsilon^x$ and $\delta_\varepsilon^x\, \bar{\delta}_\varepsilon^x = \bar{\delta}_\varepsilon^x \delta_\varepsilon^x$. (II), (III) and (IV) are consequences of these facts and theorem 8.11, with a proof similar to the one of proposition 11.3. □

Definition (38) of the coherent projection Q implies that:

$$Q^x \left(\exp \left(\sum_{i=1}^{n} a_i X_i \right)(x) \right) \;=\; \exp \left(\sum_{degX_i=1} a_i X_i \right)(x) \tag{39}$$

Therefore Q^x can be seen as a projection onto the (classical differential) geometric distribution.

Remark 11.9 *The projection Q^x has one more interesting feature: for any x and*

$$u \;=\; \exp \left(\sum_{degX_i=1} a_i X_i \right)(x)$$

we have $Q^x u = u$ and the curve

$$s \in [0,1] \mapsto \delta_s^x\, u \;=\; \exp \left(s \sum_{degX_i=1} a_i X_i \right)(x)$$

is D-horizontal and joins x and u. This will be related to the supplementary condition (B) further.

We may equally define a coherent projection which induces the dilations δ from $\tilde{\delta}$. Also, if we change the chosen normal frame with another of the same kind, we shall pass to a dilation structure which is equivalent to (U, d, δ). In conclusion, coherent projections are not geometrical objects per se, but in a natural way one may define a notion of equivalent coherent projections such that the equivalence class is geometrical, i.e. independent of the choice of a pair of particular dilation structures, each in a given equivalence class. Another way of putting this is that a class of equivalent dilation structures may be seen as a category and a coherent projection is a functor between such categories. We shall not pursue this line here.

The bottom line is that $(U, \bar{d}, \bar{\delta})$ is a dilation structure which belongs to an equivalence class which is independent on the distribution D, and also independent on the choice of parameterization ϕ. It is associated to the manifold M only. On the other hand $(U, \bar{d}, \bar{\delta})$ belongs to an equivalence class which is depending only on the distribution D and metric g on D, thus intrinsic to the sub-riemannian manifold (M, D, g). The only advantage of choosing $\bar{\delta}, \delta$ related by the normal frame $\{X_1, ..., X_n\}$ is that they are associated with a coherent projection with a simple expression.

11.2 Length functionals associated to coherent projections

Definition 11.10 *Let $(X, \bar{d}, \bar{\delta})$ be a strong dilation structure with the Radon-Nikodym property and Q a coherent projection. We define the associated distance $d : X \times X \to [0, +\infty]$ by:*

$$d(x,y) \;=\; \inf \left\{ \int_a^b \bar{d}^{c(t)}(c(t), \dot{c}(t))\; dt \;:\; c : [a,b] \to X \;\bar{d}\text{-Lipschitz}\;, \right.$$

$$\left. c(a) = x, c(b) = y, \;\; and\; \forall a.e.\; t \in [a,b] \quad Q^{c(t)} \dot{c}(t) \;=\; \dot{c}(t) \right\}$$

The relation $x \equiv y$ if $d(x,y) < +\infty$ is an equivalence relation. The space X decomposes into a reunion of equivalence classes, each equivalence class being connected by horizontal curves.

It is easy to see that d is a finite distance on each equivalence class. Indeed, from theorem 9.4 we deduce that for any $x, y \in X$ $d(x,y) \geq \bar{d}(x,y)$. Therefore $d(x,y) = 0$ implies $x = y$. The other properties of a distance are straightforward.

Later we shall give a sufficient condition (the generalized Chow condition (Cgen)) on the coherent projection Q for X to be (locally) connected by horizontal curves.

Proposition 11.11 *Suppose that X is connected by horizontal curves and (X,d) is complete. Then d is a length distance.*

Proof. Because (X,d) is complete, it is sufficient to check that d has the approximate middle property: for any $\varepsilon > 0$ and for any $x, y \in X$ there exists $z \in X$ such that

$$\max\{d(x,z), d(y,z)\} \leq \frac{1}{2}\, d(x,y) + \varepsilon$$

Given $\varepsilon > 0$, from the definition of d we deduce that there exists a horizontal curve $c : [a,b] \to X$ such that $c(a) = x$, $c(b) = y$ and $d(x,y) + 2\varepsilon \geq l(c)$ (where $l(c)$ is the length of c with respect to the distance $\bar{d}$). There exists then $\tau \in [a,b]$ such that

$$\int_a^\tau \bar{d}^{c(t)}(c(t), \dot{c}(t))\ \mathrm{d}t \ = \ \int_\tau^b \bar{d}^{c(t)}(c(t), \dot{c}(t))\ \mathrm{d}t \ = \ \frac{1}{2}\, l(c)$$

Let $z = c(\tau)$. We have then: $\max\{d(x,z), d(y,z)\} \ \leq \ \frac{1}{2}\, l(c) \ \leq \ \frac{1}{2}\, d(x,y) + \varepsilon$. Therefore d is a length distance. □

Notations concerning length functionals. The length functional associated to the distance $\bar{d}$ is denoted by $\bar{l}$. In the same way the length functional associated with $\bar{\delta}_\varepsilon^x$ is denoted by $\bar{l}_\varepsilon^x$.

We introduce the space $\mathcal{L}_\varepsilon(X,d,\delta) \subset X \times Lip([0,1], X, d)$:

$$\mathcal{L}_\varepsilon(X,d,\delta) \ = \ \{(x,c) \in X \times \mathcal{C}([0,1], X) : \ c : [0,1] \in U(x)\ ,$$

$$\delta_\varepsilon^x c \text{ is } \bar{d} - Lip, \quad Q - \text{horizontal and } Lip(\delta_\varepsilon^x c) \leq 2\varepsilon l_d(\delta_\varepsilon^x c)\big\}$$

For any $\varepsilon \in (0,1)$ we define the length functional

$$l_\varepsilon : \mathcal{L}_\varepsilon(X,d,\delta) \to [0,+\infty] \quad , \quad l_\varepsilon(x,c) \ = \ l_\varepsilon^x(c) \ = \ \frac{1}{\varepsilon}\, \bar{l}(\delta_\varepsilon^x c)$$

By theorem 9.4 we have:

$$l_\varepsilon^x(c) \ = \ \int_0^1 \frac{1}{\varepsilon}\, \bar{d}^{\delta_\varepsilon^x c(t)} \left(\delta_\varepsilon^x c(t), \frac{\bar{d}}{dt}\, (\delta_\varepsilon^x c)\, (t) \right) \ \mathrm{d}t \ =$$

$$= \ \int_0^1 \frac{1}{\varepsilon}\, \bar{d}^{\delta_\varepsilon^x c(t)} \left(\delta_\varepsilon^x c(t), \delta_\varepsilon^x\, \frac{\hat{d}_\varepsilon^x}{dt}\, c(t) \right) \ \mathrm{d}t$$

Another description of the length functional l_ε^x is the following.

Proposition 11.12 *For any $(x,c) \in \mathcal{L}_\varepsilon(X,d,\delta)$ we have*

$$l_\varepsilon^x(c) \ = \ \bar{l}_\varepsilon^x\, (Q_\varepsilon^x c)$$

Proof. Indeed, we shall use an alternate definition of the length functional. Let c be a curve such that $\delta_\varepsilon^x c$ is $\bar{d}$-Lipschitz and Q-horizontal. Then:

$$l_\varepsilon^x(c) \;=\; \sup\left\{\sum_{i=1}^{n}\frac{1}{\varepsilon}\bar{d}\left(\delta_\varepsilon^x c(t_i),\delta_\varepsilon^x c(t_{i+1})\right)\;:\;0=t_1<...<t_{n+1}=1\right\}\;=$$

$$=\;\sup\left\{\sum_{i=1}^{n}\frac{1}{\varepsilon}\bar{d}\left(\bar{\delta}_\varepsilon^x Q_\varepsilon^x c(t_i),\bar{\delta}_\varepsilon^x Q_\varepsilon^x c(t_{i+1})\right)\;:\;0=t_1<...<t_{n+1}=1\right\}\;=$$

$$=\;\bar{l}_\varepsilon^x\left(Q_\varepsilon^x c\right)$$

□

11.3 Supplementary hypotheses

Definition 11.13 *Let $(X,\bar{d},\bar{\delta})$ be a strong dilation structure and Q a coherent projection. Further is a list of supplementary hypotheses on Q:*

(A) δ_ε^x is $\bar{d}$-bilipschitz in compact sets in the following sense: for any compact set $K\subset X$ and for any $\varepsilon\in(0,1]$ there is a number $L(K)>0$ such that for any $x\in K$ and u,v sufficiently close to x we have:

$$\frac{1}{\varepsilon}\,\bar{d}\left(\delta_\varepsilon^x u,\delta_\varepsilon^x v\right)\;\leq\;L(K)\,\bar{d}(u,v)$$

(B) if $u=Q^x u$ then the curve $t\in[0,1]\mapsto Q^x\,\delta_t^x\,u\;=\;\bar{\delta}_t^x u\;=\;\delta_t^x u$ is Q-everywhere horizontal and for any $a\in[0,1]$ we have

$$\limsup_{a\to 0}\frac{\bar{l}\left(t\in[0,a]\mapsto\bar{\delta}_t^x u\right)}{\bar{d}(x,\bar{\delta}_a^x u)}\;=\;1$$

uniformly with respect to x,u in compact set K.

Condition (A), as well as the property (IV) definition 11.1, is another smoothness condition on Q with respect to the strong dilation structure $(X,\bar{d},\bar{\delta})$.

The condition (A) has several useful consequences, among them the fact that for any $\bar{d}$-Lipschitz curve c, the curve $\delta_\varepsilon^x c$ is also Lipschitz. Another consequence is that Q_ε^x is locally $\bar{d}$-Lipschitz. More precisely, for any compact set $K\subset X$ and for any $\varepsilon\in(0,1]$ there is a number $L(K)>0$ such that for any $x\in K$ and u,v sufficiently close to x we have:

$$\left(\bar{\delta}_\varepsilon^x\bar{d}\right)\left(Q_\varepsilon^x u,Q_\varepsilon^x v\right)\;\leq\;L(K)\,\bar{d}(u,v)\tag{40}$$

with the notation

$$\left(\bar{\delta}_\varepsilon^x\bar{d}\right)(u,v)\;=\;\frac{1}{\varepsilon}\,\bar{d}\left(\bar{\delta}_\varepsilon^x u,\bar{\delta}_\varepsilon^x v\right)$$

Indeed, we have:

$$\left(\bar{\delta}_\varepsilon^x\bar{d}\right)\left(Q_\varepsilon^x u,Q_\varepsilon^x v\right)\;=\;\frac{1}{\varepsilon}\,\bar{d}\left(\delta_\varepsilon^x u,\delta_\varepsilon^x v\right)\;\leq\;L(K)\,\bar{d}(u,v)$$

See the remark 11.9 for the meaning of the condition B for the case sub-riemannian geometry, where it is explained why condition B is a generalization of the fact that the "distribution" $x\mapsto Q^x U(x)$ is generated by horizontal one parameter flows.

Condition (B) will be useful later, along with the generalized Chow condition (Cgen).

12 The generalized Chow condition

Notations about words. For any set A we denote by A^* the collection of finite words $q = a_1...a_p$, $p \in \mathbb{N}$, $p > 0$. The empty word is denoted by $\emptyset$. The length of the word $q = a_1...a_p$ is $\mid q \mid = p$; the length of the empty word is 0.

The collection of words infinite at right over the alphabet A is denoted by A^ω. For any word $w \in A^\omega \cup A^*$ and any $p \in \mathbb{N}$ we denote by $[w]_p$ the finite word obtained from the first p letters of w (if $p = 0$ then $[w]_0 = \emptyset$ (in the case of a finite word q, if $p > \mid q \mid$ then $[q]_p = q$).

For any non-empty $q_1, q_2 \in A^*$ and $w \in A^\omega$ the concatenation of q_1 and q_2 is the finite word $q_1 q_2 \in A^*$ and the concatenation of q_1 and w is the (infinite) word $q_1 w \in A^\omega$. The empty word $\emptyset$ is seen both as an infinite word or a finite word and for any $q \in A^*$ and $w \in A^\omega$ we have $q\emptyset = q$ (as concatenation of finite words) and $\emptyset w = w$ (as concatenation of a finite empty word and an infinite word).

12.1 Coherent projections as transformations of words

To any coherent projection Q in a strong dilation structure $(X, \bar{d}, \bar{\delta})$ we associate a family of transformations as follows.

Definition 12.1 *For any non-empty word $w \in (0,1]^\omega$ and any $\varepsilon \in (0,1]$ we define the transformation*

$$\Psi_{\varepsilon w} : X^*_{\varepsilon w} \subset X^* \setminus \{\emptyset\} \to X^*$$

given by: for any non-empty finite word $q = xx_1...x_p \in X^*_{\varepsilon w}$ we have

$$\Psi_{\varepsilon w}(xx_1...x_p) \ = \ \Psi^1_{\varepsilon w}(x)...\Psi^{k+1}_{\varepsilon w}(xx_1...x_k)...\Psi^{p+1}_{\varepsilon w}(xx_1...x_p)$$

The functions $\Psi^k_{\varepsilon w}$ are defined by: $\Psi^1_{\varepsilon w}(x) = x$, and for any $k \geq 1$ we have

$$\Psi^{k+1}_{\varepsilon w}([q]_{k+1}) \ = \ \delta^x_{\varepsilon^{-1}} \, Q^{\delta^x_\varepsilon \, \Psi^k_{\varepsilon w}([q]_k)}_{w_k} \, \delta^x_\varepsilon \, q_{k+1} \tag{41}$$

If $w = \emptyset$ then $\Psi^k_{\varepsilon \emptyset}$ is defined as previously $\Psi^1_{\varepsilon\emptyset}(x) = x$, with the only difference that for any $k \geq 1$ we have

$$\Psi^{k+1}_{\varepsilon\emptyset}([q]_{k+1}) \ = \ \delta^x_{\varepsilon^{-1}} \, Q^{\delta^x_\varepsilon \, \Psi^k_{\varepsilon w}([q]_k)} \, \delta^x_\varepsilon \, q_{k+1}$$

The domain $X^*_{\varepsilon w} \subset X^* \setminus \{\emptyset\}$ is such that the previous definition makes sense. By using the definition of a coherent projection, we may redefine $X^*_{\varepsilon w}$ as follows: for any compact set $K \subset X$ there is $\rho = \rho(K) > 0$ such that for any $x \in K$ the word $q = xx_1...x_p \in X^*_{\varepsilon w}$ if for any $k \geq 1$ we have

$$\bar{d}\left(x_{k+1}, \Psi^k_{\varepsilon w}([q]_k)\right) \ \leq \ \rho$$

We shall explain the meaning of these transformations for $\varepsilon = 1$.

Proposition 12.2 *Suppose that condition (B) holds for the coherent projection Q. If*

$$y \ = \ \Psi^{k+1}_{1\emptyset}(xx_1...x_k)$$

then there is a Q-horizontal curve joining x and y.

Proof. By definition 12.1 for $\varepsilon = 1$ we have:

$$\Psi^1_{1w}(x) = x \quad , \quad \Psi^2_{1w}(x, x_1) \ = \ Q^x_{w_1} x_1 \quad ,$$

$$\Psi^3_{1w}(x, x_1, x_2) \ = \ Q^{Q^x_{w_1} x_1}_{w_2} x_2 \quad ...$$

Suppose now that condition (B) holds for the coherent projection Q. Then the curve $t \in [0,1] \mapsto \bar{\delta}^x_t Q^x u$ is a Q-horizontal curve joining x with $Q^x u$. Therefore by applying inductively the condition (B) we get that there is a Q-horizontal curve between $\Psi^k_{1\emptyset}(xx_1...x_{k-1})$ and $\Psi^{k+1}_{1\emptyset}(xx_1...x_k)$ for any $k > 1$ and a Q-horizontal curve joining x and $\Psi^2_{1\emptyset}(xx_1)$. □

There are three more properties of the transformations $\Psi_{\varepsilon w}$.

Proposition 12.3 *With the notations from definition 12.1 we have:*

(a) $\Psi_{\varepsilon w} \Psi_{\varepsilon \emptyset} \ = \ \Psi_{\varepsilon \emptyset}$. *Therefore we have the equality of sets:*

$$\Psi_{\varepsilon\emptyset} \left(X^*_{\varepsilon\emptyset} \cap xX^* \right) \ = \ \Psi_{\varepsilon w} \left(\Psi_{\varepsilon\emptyset} \left(X^*_{\varepsilon\emptyset} \cap xX^* \right) \right)$$

(b) $\Psi^{k+1}_{\varepsilon\emptyset}(xq_1...q_k) \ = \ \delta^x_{\varepsilon^{-1}} \Psi^{k+1}_{1\emptyset}(x\delta^x_\varepsilon q_1...\delta^x_\varepsilon q_k)$

(c) $\lim\limits_{\varepsilon \to 0} \delta^x_{\varepsilon^{-1}} \Psi^{k+1}_{1\emptyset}(x\delta^x_\varepsilon q_1...\delta^x_\varepsilon q_k) \ = \ \Psi^{k+1}_{0\emptyset}(xq_1...q_k)$ *uniformly with respect to* $x, q_1, ..., q_k$ *in compact set.*

Proof. (a) We use induction on k to prove that for any natural number k we have:

$$\Psi^{k+1}_{\varepsilon w} \left(\Psi^1_{\varepsilon\emptyset}(x)...\Psi^{k+1}_{\varepsilon\emptyset}(xq_1...q_k) \right) \ = \ \Psi^{k+1}_{\varepsilon\emptyset}(xq_1...q_k) \tag{42}$$

For $k = 0$ we have have to prove that $x = x$ which is trivial. For $k = 1$ we have to prove that

$$\Psi^2_{\varepsilon w} \left(\Psi^1_{\varepsilon\emptyset}(x) \, \Psi^2_{\varepsilon\emptyset}(xq_1) \right) \ = \ \Psi^2_{\varepsilon\emptyset}(xq_1)$$

This means:

$$\Psi^2_{\varepsilon w} \left(x \, \delta^x_{\varepsilon^{-1}} \, Q^x \delta^x_\varepsilon q_1 \right) \ = \ \delta^x_{\varepsilon^{-1}} \, Q^x_{w_1} \, \delta^x_\varepsilon \, \delta^x_{\varepsilon^{-1}} \, Q^x \, \delta^x_\varepsilon x_1 \ =$$

$$= \ \delta^x_{\varepsilon^{-1}} \, Q^x \, \delta^x_\varepsilon x_1 \ = \ \Psi^2_{\varepsilon\emptyset}(xq_1)$$

Suppose now that $l \geq 2$ and for any $k \leq l$ the relations (42) are true. Then, as previously, it is easy to check (42) for $k = l + 1$.

(b) is true by direct computation. The point (c) is a straightforward consequence of (b) and definition of coherent projections. □

Definition 12.4 *Let $N \in \mathbb{N}$ be a strictly positive natural number and $\varepsilon \in (0, 1]$. We say that $x \in X$ is* (ε, N, Q)**-nested** *in a open neighbourhood $U \subset X$ if there is $\rho > 0$ such that for any finite word $q = x_1...x_N \in X^N$ with*

$$\bar{\delta}^x_\varepsilon \bar{d} \left(x_{k+1}, \Psi^k_{\varepsilon\emptyset}([xq]_k) \right) \ \leq \ \rho$$

for any $k = 1, ..., N$, we have $q \in U^N$.

If $x \in U$ is (ε, N, Q)-nested then denote by $U(x, \varepsilon, N, Q, \rho) \subset U^N$ the collection of words $q \in U^N$ such that $\bar{\delta}^x_\varepsilon \bar{d} \left(x_{k+1}, \Psi^k_{\varepsilon\emptyset}([xq]_k) \right) \ < \ \rho$ for any $k = 1, ..., N$.

Definition 12.5 *A coherent projection Q satisfies the* **generalized Chow condition** *if:*

(Cgen) for any compact set K there are $\rho = \rho(K) > 0$, $r = r(K) > 0$, a natural number $N = N(Q, K)$ and a function $F(\eta) = \mathcal{O}(\eta)$ such that for any $x \in K$ and $\varepsilon \in (0, 1]$ there are neighbourhoods $U(x)$, $V(x)$ such that any $x \in K$ is (ε, N, Q)-nested in $U(x)$, $B(x, r, \bar{\delta}_\varepsilon^x \bar{d}) \subset V(x)$ and such that the mapping

$$x_1 ... x_N \in U(x, N, Q, \rho) \mapsto \Psi_{\varepsilon\emptyset}^{N+1}(x x_1 ... x_N)$$

is surjective from $U(x, \varepsilon, N, Q, \rho)$ to $V(x)$. Moreover for any $z \in V(x)$ there exist $y_1, ... y_N \in U(x, \varepsilon, N, Q, \rho)$ such that $z = \Psi_{\varepsilon\emptyset}^{N+1}(x y_1, ... y_N)$ and for any $k = 0, ..., N-1$ we have

$$\delta_\varepsilon^x \bar{d}\left(\Psi_{\varepsilon\emptyset}^{k+1}(x y_1 ... y_k), \Psi_{\varepsilon\emptyset}^{k+2}(x y_1 ... y_{k+1})\right) \ \leq \ F(\delta_\varepsilon^x \bar{d}(x, z))$$

Condition (Cgen) is inspired from lemma 1.40 Folland-Stein [19]. If the coherent projection Q satisfies also (A) and (B) then in the space $(U(x), \bar{\delta}_\varepsilon^x)$, with coherent projection $\hat{Q}_{\varepsilon,\cdot}^{x,\cdot}$, we can join any two sufficiently close points by a sequence of at most N horizontal curves. Moreover there is a control on the length of these curves via condition (B) and condition (Cgen); in sub-riemannian geometry the function F is of the type $F(\eta) = \eta^{1/m}$ with m positive natural number.

Definition 12.6 *Suppose that the coherent projection Q satisfies conditions (A), (B) and (Cgen). Let us consider $\varepsilon \in (0, 1]$ and $x, y \in K$, K compact in X. With the notations from definition 12.5, suppose that there are numbers $N = N(Q, K)$, $\rho = \rho(Q, K) > 0$ and words $x_1 ... x_N \in U(x, \varepsilon, N, Q, \rho)$ such that*

$$y = \Psi_{\varepsilon\emptyset}^{N+1}(x x_1 ... x_N)$$

To these data we associate a **short curve** *joining x and y, $c : [0, N] \to X$ defined by: for any $t \in [0, N]$ then let $k = [t]$, where $[b]$ is the integer part of the real number b. We define the short curve by*

$$c(t) \ = \ \bar{\delta}_{\varepsilon, t+N-k}^{x, \Psi_{\varepsilon\emptyset}^{k+1}(x x_1 ... x_k)} Q^{\Psi_{\varepsilon\emptyset}^{k+1}(x x_1 ... x_k)} x_{k+1}$$

Any short curve joining x and y is a increasing linear reparameterization of a curve c described previously.

12.2 The candidate tangent space

Let $(X, \bar{d}, \bar{\delta})$ be a strong dilation structure and Q a coherent projection. Then we have the induced dilations

$$\mathring{\delta}_\mu^{x,u} v \ = \ \Sigma^x(u, \delta_\mu^x \Delta^x(u, v))$$

and the induced projection

$$\mathring{Q}_\mu^{x,u} v \ = \ \Sigma^x(u, Q_\mu^x \Delta^x(u, v))$$

For any curve $c : [0, 1] \to U(x)$ which is $\mathring{\delta}^x$-derivable and $\mathring{Q}^x$-horizontal almost everywhere:

$$\frac{\mathring{d}^x}{dt} c(t) \ = \ \mathring{Q}^{x,u} \frac{\mathring{d}^x}{dt} c(t)$$

we define the length

$$l^x(c) \ = \ \int_0^1 \bar{d}^x \left(x, \Delta^x(c(t), \frac{\mathring{d}^x}{dt} c(t))\right) \ \mathrm{d}t$$

and the distance function:

$$\mathring{d}^x(u, v) \ = \ \inf \Big\{ l^x(c) \ : \ c : [0, 1] \to U(x) \ \text{ is } \mathring{\delta}^x\text{-derivable},$$

$$\text{and } \mathring{Q}^x\text{-horizontal a.e. }, c(0)=u, c(1)=v\Big\}$$

We want to prove that $(U(x), \mathring{d}^x, \mathring{\delta}^x)$ is a strong dilation structure and $\mathring{Q}^x$ is a coherent projection. For this we need first the following proposition.

Proposition 12.7 *The curve $c : [0,1] \to U(x)$ is $\mathring{\delta}^x$-derivable, $\mathring{Q}^x$-horizontal almost everywhere, and $l^x(c) < +\infty$ if and only if the curve $Q^x c$ is $\bar{\delta}^x$-derivable almost everywhere and $\bar{l}^x(Q^x c) < +\infty$. Moreover, we have*

$$\bar{l}^x(Q^x c) \;=\; l^x(c)$$

Proof. The curve c is $\mathring{Q}^x$-horizontal almost everywhere if and only if for almost any $t \in [0,1]$ we have

$$Q^x\,\Delta^x(c(t), \frac{\mathring{d}^x}{dt}c(t)) \;=\; \Delta^x(c(t), \frac{\mathring{d}^x}{dt}c(t))$$

We shall prove that c is $\mathring{Q}^x$-horizontal is equivalent with

$$\Theta^x(c(t), \frac{\mathring{d}^x}{dt}c(t)) \;=\; \frac{\bar{d}^x}{dt}\left(Q^x c\right)(t) \tag{43}$$

Indeed, (43) is equivalent with

$$\lim_{\varepsilon\to 0}\bar{\delta}^x_{\varepsilon^{-1}}\bar{\Delta}^x(Q^x c(t), Q^x c(t+\varepsilon)) \;=\; \bar{\Delta}^x(Q^x c(t), \Theta^x(c(t), \frac{\mathring{d}^x}{dt}c(t)))$$

which is equivalent with

$$\lim_{\varepsilon\to 0}\bar{\delta}^x_{\varepsilon^{-1}}\bar{\Delta}^x(Q^x c(t), Q^x c(t+\varepsilon)) \;=\; \Delta^x(c(t), \frac{\mathring{d}^x}{dt}c(t))$$

But this is equivalent with:

$$\lim_{\varepsilon\to 0}\bar{\delta}^x_{\varepsilon^{-1}}\bar{\Delta}^x(Q^x c(t), Q^x c(t+\varepsilon)) \;=\; \lim_{\varepsilon\to 0}\delta^x_{\varepsilon^{-1}}\Delta^x(c(t), c(t+\varepsilon)) \tag{44}$$

The horizontality condition for the curve c can be written as:

$$\lim_{\varepsilon\to 0} Q^x\delta^x_{\varepsilon^{-1}}\Delta^x(c(t), c(t+\varepsilon)) \;=\; \lim_{\varepsilon\to 0}\delta^x_{\varepsilon^{-1}}\Delta^x(c(t), c(t+\varepsilon))$$

We use now the properties of Q^x in the left hand side of the previous equality:

$$Q^x\delta^x_{\varepsilon^{-1}}\Delta^x(c(t), c(t+\varepsilon)) \;=\; \bar{\delta}^x_{\varepsilon^{-1}}Q^x\Delta^x(c(t), c(t+\varepsilon)) \;=$$

$$=\; \bar{\delta}^x_{\varepsilon^{-1}}\bar{\Delta}^x(Q^x c(t), Q^x c(t+\varepsilon))$$

thus after taking the limit as $\varepsilon \to 0$ we prove that the limit

$$\lim_{\varepsilon\to 0}\bar{\delta}^x_{\varepsilon^{-1}}\bar{\Delta}^x(Q^x c(t), Q^x c(t+\varepsilon))$$

exists and we obtain:

$$\lim_{\varepsilon\to 0}\delta^x_{\varepsilon^{-1}}\Delta^x(c(t), c(t+\varepsilon)) \;=\; \lim_{\varepsilon\to 0}\bar{\delta}^x_{\varepsilon^{-1}}\bar{\Delta}^x(Q^x c(t), Q^x c(t+\varepsilon))$$

This last equality is the same as (44), which is equivalent with (43).

As a consequence we obtain the following equality, for almost any $t \in [0,1]$:

$$\bar{d}^x\left(x, \Delta^x(c(t), \frac{\mathring{d}^x}{dt}c(t))\right) \;=\; \bar{\Delta}^x(Q^x c(t), \frac{\bar{d}^x}{dt}\left(Q^x c\right)(t)) \tag{45}$$

This implies that $Q^x c$ is absolutely continuous and by theorem 6.8, as in the proof of theorem 9.4 (but without using the Radon-Nikodym property property, because we already know that $Q^x c$ is derivable a.e.), we obtain the following formula for the length of the curve $Q^x c$:

$$\bar{l}^x(Q^x c) \;=\; \int_0^1 \bar{d}^x\left(x, , \bar{\Delta}^x(Q^x c(t), \frac{\bar{d}^x}{dt}\left(Q^x c\right)(t))\right)\; \mathrm{d}t$$

But we have also:

$$l^x(c) \;=\; \int_0^1 \bar{d}^x\left(x, \Delta^x(c(t), \frac{\mathring{d}^x}{dt}c(t))\right)\; \mathrm{d}t$$

By (45) we obtain $\bar{l}^x(Q^x c) \,=\, l^x(c)$. □

Proposition 12.8 *If $(X, \bar{d}, \bar{\delta})$ is a strong dilation structure, Q is a coherent projection and $\mathring{d}^x$ is finite then the triple $(U(x), \Sigma^x, \delta^x)$ is a normed conical group, with the norm induced by the left-invariant distance $\mathring{d}^x$.*

Proof. The fact that $(U(x), \Sigma^x, \delta^x)$ is a conical group comes directly from the definition 11.1 of a coherent projection. Indeed, it is enough to use proposition 11.3 (c) and the formalism of binary decorated trees in [5] section 4 (or theorem 11 [5]), in order to reproduce the part of the proof of theorem 10 (p.87-88) in that paper, concerning the conical group structure. There is one small subtlety though. In the proof of theorem 7.6(a) the same modification of proof has been done starting from the axiom A4+, namely the existence of the uniform limit $\lim\limits_{\varepsilon \to 0} \Sigma^x_\varepsilon(u,v) \,=\, \Sigma^x(u,v)$. Here we need first to prove this limit, in a similar way as in the corollary 9 [5]. We shall use for this the distance $\mathring{d}^x$ instead of the distance in the metric tangent space of (X,d) at x denoted by d^x (which is not yet proven to exist). The distance $\mathring{d}^x$ is supposed to be finite by hypothesis. Moreover, by its definition and proposition 12.7 we have

$$\mathring{d}^x(u,v) \;\geq\; \bar{d}^x(u,v)$$

therefore the distance $\mathring{d}^x$ is non degenerate. By construction this distance is also left invariant with respect to the group operation $\Sigma^x(\cdot,\cdot)$. Therefore we may repeat the proof of corollary 9 [5] and obtain the result that A4+ is true for (X,d,δ).

What we need to prove next is that $\mathring{d}^x$ induces a norm on the conical group $(U(x), \Sigma^x, \delta^x)$. For this it is enough to prove that

$$\mathring{d}^x(\mathring{\delta}^{x,u}_\mu v, \mathring{\delta}^{x,u}_\mu w) \;=\; \mu\, \mathring{d}^x(v,w) \tag{46}$$

for any $v, w \in U(x)$. This is a direct consequence of relation (45) from the proof of the proposition 12.7. Indeed, by direct computation we get that for any curve c which is $\mathring{Q}^x$-horizontal a.e. we have:

$$l^x(\mathring{\delta}^{x,u}_\mu c) \;=\; \int_0^1 \bar{d}^x\left(x, \Delta^x\left(\mathring{\delta}^{x,u}_\mu c(t), \frac{\mathring{d}^x}{dt}\left(\mathring{\delta}^{x,u}_\mu c\right)(t)\right)\right)\; \mathrm{dt} \;=$$

$$= \int_0^1 \bar{d}^x\left(x, \delta^x_\mu \Delta^x\left(c(t), \frac{\mathring{d}^x}{dt}c(t)\right)\right)\; \mathrm{dt}$$

But c is $\mathring{Q}^x$-horizontal a.e., which implies, via (45), that

$$\delta_\mu^x \Delta^x \left(c(t), \frac{\mathring{d}^x}{dt} c(t) \right) \ = \ \bar{\delta}_\mu^x \Delta^x \left(c(t), \frac{\mathring{d}^x}{dt} c(t) \right)$$

therefore we have

$$l^x(\mathring{\delta}_\mu^{x,u} c) \ = \ \int_0^1 \bar{d}^x \left(x, \bar{\delta}_\mu^x \Delta^x \left(c(t), \frac{\mathring{d}^x}{dt} c(t) \right) \right) \ \mathrm{dt} \ = \ \mu \, l^x(c)$$

This implies (46), therefore the proof is done. □

Theorem 12.9 *If the generalized Chow condition (Cgen) and condition (B) are true then $(U(x), \Sigma^x, \delta^x)$ is local conical group which is a neighbourhood of the neutral element of a Carnot group generated by $Q^x U(x)$.*

Proof. For any $\varepsilon \in (0,1]$, as a consequence of proposition 11.6 we can put the recurrence relations (41) in the form:

$$\Psi_{\varepsilon w}^{k+1}([q]_{k+1}) \ = \ \Sigma_\varepsilon^x \left(\Psi_{\varepsilon w}^k([q]_k), Q_{w_k}^{\delta_\varepsilon^x \Psi_{\varepsilon w}^k([q]_k)} \, \Delta_\varepsilon^x \left(\Psi_{\varepsilon w}^k([q]_k), q_{k+1} \right) \right) \tag{47}$$

This recurrence relation allows us to prove by induction that for any k the limit

$$\Psi_w^k([q]_k) \ = \ \lim_{\varepsilon \to 0} \Psi_{\varepsilon w}^k([q]_k)$$

exists and it satisfies the recurrence relation:

$$\Psi_{0w}^{k+1}([q]_{k+1}) \ = \ \Sigma^x \left(\Psi_{0w}^k([q]_k), Q_{w_k}^x \, \Delta^x \left(\Psi_{0w}^k([q]_k), q_{k+1} \right) \right) \tag{48}$$

and the initial condition $\Psi_{0w}^1(x) = x$. We pass to the limit in the generalized Chow condition (Cgen) and we thus obtain that a neighbourhood of the neutral element x is (algebraically) generated by $Q^x U(x)$. Then the distance $\mathring{d}^x$. Therefore by proposition 12.8 $(U(x), \Sigma^x, \delta^x)$ is a normed conical group generated by $Q^x U(x)$.

Let $c : [0,1] \to U(x)$ be the curve $c(t) = \delta_t^x u$, with $u \in Q^x U(x)$. Then we have $Q^x c(t) = c(t) = \bar{\delta}_t^x u$. From condition (B) we get that c is $\bar{\delta}$-derivable at $t = 0$. A short computation of this derivative shows that:

$$\frac{d\bar{\delta}}{dt} c(0) \ = \ u$$

Another easy computation shows that the curve c is $\bar{\delta}^x$-derivable if and only if the curve c is $\bar{\delta}$-derivable at $t = 0$, which is true, therefore c is $\bar{\delta}^x$-derivable, in particular at $t = 0$. Moreover, the expression of the $\bar{\delta}^x$-derivative of c shows that c is also Q^x-everywhere horizontal (compare with the remark 11.9). We use the proposition 12.7 and relation (43) from its proof to deduce that $c = Q^x c$ is $\mathring{\delta}^x$-derivable at $t = 0$, thus for any $u \in Q^x U(x)$ and small enough $t, \tau \in (0,1)$ we have

$$\mathring{\delta}_{t+\tau}^{x,x} u \ = \ \bar{\Sigma}^x(\bar{\delta}_t^x u, \bar{\delta}_\tau^x u) \tag{49}$$

By the previous proposition 12.8 and corollary 6.3 [6], the normed conical group $(U(x), \Sigma^x, \delta^x)$ is in fact locally a homogeneous group, i.e. a simply connected Lie group which admits a positive graduation given by the eigenspaces of δ^x. Indeed, corollary 6.3 [5] is originally about strong dilation structures, but the generalized Chow condition implies that the distances d, $\bar{d}$ and $\mathring{d}^x$ induce the same uniformity, which, along with proposition 12.8, are the only things needed for the proof of this corollary. The conclusion of corollary 6.3 [6] therefore is true, that is $(U(x), \Sigma^x, \delta^x)$ is locally a homogeneous group. Moreover it is locally Carnot if and only if on the generating space $Q^x U(x)$ any dilation $\mathring{\delta}_\varepsilon^{x,x} u \ = \ \bar{\delta}_\varepsilon^x$ is linear in ε. But this is true, as shown by relation (49). This ends the proof. □

12.3 Coherent projections induce length dilation structures

Theorem 12.10 *If $(X, \bar{d}, \bar{\delta})$ is a tempered strong dilation structure, has the Radon-Nikodym property and Q is a coherent projection, which satisfies (A), (B), (Cgen) then (X, d, δ) is a length dilation structure.*

Proof. We shall prove that:

(a) for any function $\varepsilon \in (0,1) \mapsto (x_\varepsilon, c_\varepsilon) \in \mathcal{L}_\varepsilon(X, d, \delta)$ which converges to (x, c) as $\varepsilon \to 0$, with $c : [0,1] \to U(x)$ $\mathring{\delta}^x$-derivable and $\mathring{Q}^x$-horizontal almost everywhere, we have:

$$l^x(c) \ \leq \ \liminf_{\varepsilon \to 0} l^{x_\varepsilon}(c_\varepsilon)$$

(b) for any sequence $\varepsilon_n \to 0$ and any (x, c), with $c : [0,1] \to U(x)$ $\mathring{\delta}^x$-derivable and $\mathring{Q}^x$-horizontal almost everywhere, there is a recovery sequence $(x_n, c_n) \in \mathcal{L}_{\varepsilon_n}(X, d, \delta)$ such that

$$l^x(c) \ = \ \lim_{n \to \infty} l^{x_n}(c_n)$$

Proof of (a). This is a consequence of propositions 12.7, 11.12 and definition 11.1 of a coherent projection. With the notations from (a) we see that we have to prove

$$l^x(c) \ = \ \bar{l}^x(Q^x c) \ \leq \ \liminf_{\varepsilon \to 0} \bar{l}^{x_\varepsilon}(Q^{x_\varepsilon}_\varepsilon c_\varepsilon)$$

This is true because $(X, \bar{d}, \bar{\delta})$ is a tempered dilation structure and because of condition (A). Indeed from the fact that $(X, \bar{d}, \bar{\delta})$ is tempered and from (40) (which is a consequence of condition (A)) we deduce that Q_ε is uniformly continuous on compact sets in a uniform way: for any compact set $K \subset X$ there is are constants $L(K) > 0$ (from (A)) and $C > 0$ (from the tempered condition) such that for any $\varepsilon \in (0,1]$, any $x \in K$ and any u, v sufficiently close to x we have:

$$\bar{d}\,(Q^x_\varepsilon u, Q^x_\varepsilon v) \ \leq \ C \ \left(\bar{\delta}^x_\varepsilon \bar{d}\right)(Q^x_\varepsilon u, Q^x_\varepsilon v) \ \leq \ C\, L(K)\, \bar{d}(u, v)$$

The sequence Q^x_ε uniformly converges to Q^x as ε goes to 0, uniformly with respect to x in compact sets. Therefore if $(x_\varepsilon, c_\varepsilon) \in \mathcal{L}_\varepsilon(X, d, \delta)$ converges to (x, c) then $(x_\varepsilon, Q^{x_\varepsilon}_\varepsilon c_\varepsilon) \in \mathcal{L}_\varepsilon(X, \bar{d}, \bar{\delta})$ converges to $(x, Q^x c)$. Use now the fact that by corollary 10.4 $(X, \bar{d}, \bar{\delta})$ is a length dilation structure. The proof is done.

Proof of (b). We have to construct a recovery sequence. We are doing this by discretization of $c : [0, L] \to U(x)$. Recall that c is a curve which is $\mathring{\delta}^x$-derivable a.e. and $\mathring{Q}^x$-horizontal, that is for almost every $t \in [0, L]$ the limit

$$u(t) \ = \ \lim_{\mu \to 0} \delta^x_{\mu^{-1}}\, \Delta^x(c(t), c(t+\mu))$$

exists and $Q^x\, u(t) = u(t)$. Moreover we may suppose that for almost every t we have $\bar{d}^x(x, u(t)) \leq 1$ and $\bar{l}^x(c) \leq L$.

There are functions $\omega^1, \omega^2 : (0, +\infty) \to [0, +\infty)$ with $\lim_{\lambda \to 0} \omega^i(\lambda) = 0$, with the following property: for any $\lambda > 0$ sufficiently small there is a division $A_\lambda = \{0 < t_0 < ... < t_P < L\}$ such that

$$\frac{\lambda}{2} \ \leq \ \min\left\{ \frac{t_0}{t_1 - t_0}, \frac{L - t_P}{t_P - t_{P-1}}, t_k - t_{k-1} \,:\, k = 1, ..., P \right\} \tag{50}$$

$$\lambda \ \geq \ \max\left\{ \frac{t_0}{t_1 - t_0}, \frac{L - t_P}{t_P - t_{P-1}}, t_k - t_{k-1} \,:\, k = 1, ..., P \right\} \tag{51}$$

and such that $u(t_k)$ exists for any $k = 1, ..., P$ and

$$\mathring{d}^x(c(0), c(t_0)) \leq t_0 \leq \lambda^2 \tag{52}$$

$$\mathring{d}^x(c(L), c(t_P)) \leq L - t_P \leq \lambda^2 \tag{53}$$

$$\mathring{d}^x(u(t_{k-1}), \Delta^x(c(t_{k-1}), c(t_k)) \leq (t_k - t_{k-1})\, \omega^1(\lambda) \tag{54}$$

$$| \int_0^L \bar{d}^x(x, u(t))\, \mathrm{d}t \; - \; \sum_{k=0}^{P-1} (t_{k+1} - t_k)\, \bar{d}^x(x, u(t_k)) \,| \leq \omega^2(\lambda) \tag{55}$$

Indeed (52), (53) are a consequence of the fact that c is $\mathring{d}^x$-Lipschitz, (54) is a consequence of Egorov theorem applied to

$$f_\mu(t) \; = \; \delta^x_{\mu^{-1}}\, \Delta^x(c(t), c(t+\mu))$$

and (55) comes from the definition of the integral

$$l(c) \; = \; \int_0^L \bar{d}^x(x, u(t))\, \mathrm{d}t$$

For each λ we shall choose $\varepsilon = \varepsilon(\lambda)$ and we shall construct a curve c_λ with the properties:

(i) $(x, c_\lambda) \in \mathcal{L}_{\varepsilon(\lambda)}(X, d, \delta)$

(ii) $\lim\limits_{\lambda \to 0} l^x_{\varepsilon(\lambda)}(c_\lambda) \; = \; l^x(c).$

At almost every t the point $u(t)$ represents the velocity of the curve c seen as the the left translation of $\frac{\mathring{d}^x}{dt} c(t)$ by the group operation $\Sigma^x(\cdot, \cdot)$ to x (which is the neutral element for the mentioned operation). The derivative (with respect to $\mathring{\delta}^x$) of the curve c at t is

$$y(t) \; = \; \Sigma^x(c(t), u(t))$$

Let us take $\varepsilon > 0$, arbitrary for the moment. We shall use the points of the division A_λ and for any $k = 0, ..., P-1$ we shall define the point:

$$y^\varepsilon_k \; = \; \hat{Q}^{x, c(t_k)}_\varepsilon\, \Sigma^x_\varepsilon(c(t_k), u(t_k)) \tag{56}$$

Thus y^ε_k is obtained as the "projection" by $\hat{Q}^{x, c(t_k)}_\varepsilon$ of the "approximate left translation" $\Sigma^x_\varepsilon(c(t_k), \cdot)$ by $c(t_k)$ of the velocity $u(t_k)$. Define also the point:

$$y_k \; = \; \Sigma^x(c(t_k), u(t_k))$$

By construction we have:

$$y^\varepsilon_k \; = \; \hat{Q}^{x, c(t_k)}_\varepsilon\, y^\varepsilon_k \tag{57}$$

and by computation we see that y^ε_k can be expressed as:

$$y^\varepsilon_k \; = \; \delta^x_{\varepsilon^{-1}}\, Q^{\delta^x_\varepsilon c(t_k)}\, \delta^{\delta^x_\varepsilon c(t_k)}_\varepsilon\, u(t_k) \; = \tag{58}$$

$$= \; \Sigma^x_\varepsilon(c(t_k), Q^{\delta^x_\varepsilon c(t_k)}\, u(t_k)) \; = \; \delta^x_{\varepsilon^{-1}}\, \bar{\delta}^{\delta^x_\varepsilon c(t_k)}_\varepsilon\, Q^{\delta^x_\varepsilon c(t_k)}\, u(t_k)$$

Let us define the curve

$$c^\varepsilon_k(s) \; = \; \hat{\delta}^{x, c(t_k)}_{\varepsilon, s}\, y^\varepsilon_k \quad , \quad s \in [0, t_{k+1} - t_k] \tag{59}$$

which is a $\hat{Q}^x_\varepsilon$-horizontal curve (by supplementary hypothesis (B)) which joins $c(t_k)$ with the point

$$z^\varepsilon_k \;=\; \hat{\delta}^{x,c(t_k)}_{\varepsilon,t_{k+1}-t_k}\, y^\varepsilon_k \tag{60}$$

The point z^ε_k is an approximation of the point

$$z_k \;=\; \mathring{\delta}^{x,c(t_k)}_{t_{k+1}-t_k} y_k$$

We shall also consider the curve

$$c_k(s) \;=\; \mathring{\delta}^{x,c(t_k)}_s\, y_k \quad , \quad s \in [0, t_{k+1}-t_k] \tag{61}$$

There is a short curve g^ε_k which joins z^ε_k with $c(t_{k+1})$, according to condition (Cgen). Indeed, for ε sufficiently small the points $\delta^x_\varepsilon\, z^\varepsilon_k$ and $\delta^x_\varepsilon\, c(t_{k+1})$ are sufficiently close.

Finally, take g^ε_0 and g^ε_{P+1} "short curves" which join $c(0)$ with $c(t_0)$ and $c(t_P)$ with $c(L)$ respectively.

Correspondingly, we can find short curves g_k (in the geometry of the dilation structure $(U(x), \mathring{d}^x, \mathring{\delta}^x, \mathring{Q}^x)$) joining z_k with $c(t_{k+1})$, which are the uniform limit of the short curves g^ε_k as $\varepsilon \to 0$. Moreover this convergence is uniform with respect to k (and λ). Indeed, these short curves are made by N curves of the type $s \mapsto \hat{\delta}^{x,u_\varepsilon}_{\varepsilon,s} v_\varepsilon$, with $\hat{Q}^{x,u_\varepsilon} v_\varepsilon \;=\; v_\varepsilon$. Also, the short curves g_k are made respectively by N curves of the type $s \mapsto \mathring{\delta}^{x,u}_s v$, with $\mathring{Q}^{x,u} v \;=\; v$. Therefore we have:

$$\bar{d}(\mathring{\delta}^{x,u}_s v, \hat{\delta}^{x,u_\varepsilon}_{\varepsilon,s} y^\varepsilon_k) \;=$$

$$=\; \bar{d}(\Sigma^x(u, \bar{\delta}^x_s \Delta^x(u,v)), \Sigma^x_\varepsilon(u_\varepsilon, \bar{\delta}^{\delta^x_\varepsilon u_\varepsilon}_s \Delta^x_\varepsilon(u_\varepsilon, v_\varepsilon)))$$

By an induction argument on the respective ends of segments forming the short curves, using the axioms of coherent projections, we get the result.

By concatenation of all these curves we get two new curves:

$$c^\varepsilon_\lambda \;=\; g^\varepsilon_0 \left(\prod_{k=0}^{P-1} c^\varepsilon_k\, g^\varepsilon_k \right) g^\varepsilon_{P+1}$$

$$c_\lambda \;=\; g_0 \left(\prod_{k=0}^{P-1} c_k\, g_k \right) g_{P+1}$$

From the previous reasoning we get that as $\varepsilon \to 0$ the curve c^ε_λ uniformly converges to c_λ, uniformly with respect to λ.

By theorem 12.9, specifically from relation (49) and considerations below, we notice that for any $u \;=\; Q^x u$ the length of the curve $s \mapsto \delta^x_s u$ is:

$$l^x(s \in [0,a] \mapsto \delta^x_s u) \;=\; a\, \bar{d}^x(x,u)$$

From here and relations (52), (53), (54), (55) we get that

$$l^x(c) \;=\; \lim_{\lambda \to 0} l^x(c_\lambda) \tag{62}$$

Condition (B) and the fact that $(X, \bar{d}, \bar{\delta})$ is tempered imply that there is a positive function $\omega^3(\varepsilon) = \mathcal{O}(\varepsilon)$ such that

$$|\; l^x_\varepsilon(c^\varepsilon_\lambda) - l^x(c_\lambda) \;| \leq\; \frac{\omega^3(\varepsilon)}{\lambda} \tag{63}$$

This is true because if $v\,\hat{Q}_\varepsilon^{x,u}v$ then $\delta_\varepsilon^x v = Q^{\delta_\varepsilon^x u}\delta_\varepsilon^x v$, therefore by condition (B)

$$\frac{l_\varepsilon^x(s\in[0,a]\mapsto\hat{\delta}_{\varepsilon,s}^{x,u}v)}{\delta_\varepsilon^x\bar{d}(u,v)} = \frac{\bar{l}(s\in[0,a]\mapsto\bar{\delta}_s^{\delta_\varepsilon^x u}\delta_\varepsilon^x v)}{\bar{d}(\delta_\varepsilon^x u,\delta_\varepsilon^x v)} \leq \mathcal{O}(\varepsilon)+1$$

Since each short curve is made by N segments and the division A_λ is made by $1/\lambda$ segments, the relation (63) follows.

We shall choose now $\varepsilon(\lambda)$ such that $\omega^3(\varepsilon(\lambda))\leq\lambda^2$ and we define:

$$c_\lambda = c_\lambda^{\varepsilon(\lambda)}$$

These curves satisfy the properties (i), (ii). Indeed (i) is satisfied by construction and (ii) follows from the choice of $\varepsilon(\lambda)$, uniform convergence of c_λ^ε to c_λ, uniformly with respect to λ, and relations (63), and (62). □

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