

# Control of Nonholonomic Systems and Sub-Riemannian Geometry

Frédéric JEAN<sup>\*†</sup>

Lectures given at the CIMPA School “Géométrie  
sous-riemannienne”, Beirut, Lebanon, 2012

## Contents

<b>1</b>	<b>Geometry of nonholonomic systems</b>	<b>2</b>
1.1	Nonholonomic systems . . . . .	2
1.2	Sub-Riemannian distances . . . . .	2
<b>2</b>	<b>Controllability</b>	<b>5</b>
2.1	The Chow-Rashevsky Theorem . . . . .	5
2.2	Topological structure of $(M, d)$ . . . . .	7
<b>3</b>	<b>First-order approximations</b>	<b>8</b>
3.1	Nonholonomic orders . . . . .	9
3.2	Privileged coordinates . . . . .	13
3.3	Nilpotent approximation . . . . .	20
3.4	Distance estimates . . . . .	23
3.5	Approximate motion planning . . . . .	27
<b>4</b>	<b>Tangent structure</b>	<b>28</b>
4.1	Metric tangent space . . . . .	28
4.2	Desingularization and uniform distance estimate . . . . .	32

---

<sup>\*</sup>ENSTA ParisTech, UMA, 32, bd Victor, 75015 Paris, France and Team GECO, INRIA Saclay – Île-de-France, [frederic.jean@ensta-paristech.fr](mailto:frederic.jean@ensta-paristech.fr)

<sup>†</sup>This work was supported by the ANR project *GCM*, program “Blanche”, project number NT09.504490, and by the Commission of the European Communities under the 7th Framework Programme Marie Curie Initial Training Network (FP7-PEOPLE-2010-ITN), project SADCO, contract number 264735.

Nonholonomic systems are control systems which depend linearly on the control. Their underlying geometry is the sub-Riemannian geometry, which plays for these systems the same role as Euclidean geometry does for linear systems. In particular the usual notions of approximations at the first order, that are essential for control purposes, have to be defined in terms of this geometry. The aim of these notes is to present these notions of approximation and their link with the metric tangent structure in sub-Riemannian geometry.

# 1 Geometry of nonholonomic systems

Let  $M$  be a smooth  $n$ -dimensional manifold.

## 1.1 Nonholonomic systems

A *nonholonomic system* on  $M$  is a control system which is of the form

$$(\Sigma) \quad \dot{q} = \sum_{i=1}^m u_i X_i(q), \quad q \in M, \quad u = (u_1, \dots, u_m) \in \mathbb{R}^m, \quad (1)$$

where  $X_1, \dots, X_m$  are  $C^\infty$  vector fields on  $M$ .

A path  $\gamma : [0, T] \rightarrow M$  is a *trajectory of (1)* if there exists a function  $u(\cdot) \in L^1([0, T], \mathbb{R}^m)$ , called a *control law* associated to  $\gamma$ , such that  $\gamma$  is solution of the ordinary differential equation:

$$\dot{q}(t) = \sum_{i=1}^m u_i(t) X_i(q(t)), \quad t \in [0, T].$$

Equivalently, a trajectory is an absolutely continuous path  $\gamma$  on  $M$  such that  $\dot{\gamma}(t) \in \Delta(\gamma(t))$  for almost every  $t \in [0, T]$ , where we have set, for every  $q \in M$ ,

$$\Delta(q) = \text{span} \{X_1(q), \dots, X_m(q)\} \subset T_q M. \quad (2)$$

## 1.2 Sub-Riemannian distances

A nonholonomic system induces a distance on  $M$  in the following way. We first define the *sub-Riemannian metric* associated to (1) to be the function  $g : TM \rightarrow \overline{\mathbb{R}}$  given by: for every  $q \in M$  and  $v \in T_q M$ ,

$$g(q, v) = \inf \left\{ u_1^2 + \dots + u_m^2 : \sum_{i=1}^m u_i X_i(q) = v \right\}, \quad (3)$$

where we adopt the convention that  $\inf \emptyset = +\infty$ . This function  $g$  is smooth and satisfies:

- $g(q, v) = +\infty$  if  $v \notin \Delta(q)$ ,
- $g$  restricted to  $\Delta(q)$  is a positive definite quadratic form.

Such a metric allows to define a distance in the same way as in Riemannian geometry. The *length* of an absolutely continuous path  $\gamma(t)$ ,  $t \in [0, T]$ , is

$$\text{length}(\gamma) = \int_0^T \sqrt{g(\gamma(t), \dot{\gamma}(t))} dt.$$

Note that only trajectories of (1) may have a finite length. Finally, the *sub-Riemannian distance* on  $M$  associated to the nonholonomic system (1) is defined by

$$d(p, q) = \inf \text{length}(\gamma),$$

where the infimum is taken over all absolutely continuous path  $\gamma$  joining  $p$  to  $q$  (the fact that  $d$  is actually a distance will be proved in subsection 2.2). In particular, if no trajectory joins  $p$  to  $q$ ,  $d(p, q) = +\infty$ .

The distance  $d$  so defined does not always meet the classical notion of sub-Riemannian distance arising from a sub-Riemannian manifold. Let us recall the latter definition.

A *sub-Riemannian manifold*  $(M, D, g_R)$  is a smooth manifold  $M$  endowed with a *sub-Riemannian structure*  $(D, g_R)$ , where:

- $D$  is a distribution on  $M$ , that is a subbundle of  $TM$ ;
- $g_R$  is a Riemannian metric on  $D$ , that is a smooth function  $g_R : \Delta \rightarrow \mathbb{R}$  which restrictions to  $D(q)$  are positive definite quadratic forms.

The *sub-Riemannian metric* associated to  $(D, g_R)$  is the function  $g_{SR} : TM \rightarrow \overline{\mathbb{R}}$  given by

$$g_{SR}(q, v) = \begin{cases} g_R(q, v) & \text{if } v \in D(q), \\ +\infty & \text{otherwise.} \end{cases} \quad (4)$$

The *sub-Riemannian distance*  $d_{SR}$  on  $(M, D, g_R)$  is then defined from  $g_{SR}$  as above.

What is the difference between the two constructions, that is between the definitions (3) and (4) of a sub-Riemannian metric?

Consider a *sub-Riemannian structure*  $(D, g_R)$ . Locally, on some open subset  $U$ , there exist vector fields  $X_1, \dots, X_m$  which values at each point  $q \in U$  form an orthonormal basis of  $D(q)$  for the quadratic form  $g_R$ ; the metric  $g_{SR}$  associated to  $(D, g_R)$  then coincides with the metric  $g$  associated to  $X_1, \dots, X_m$ . Thus, locally, there is a one-to-one correspondence between

sub-Riemannian structures and nonholonomic systems for which the rank of  $\Delta(q) = \text{span}\{X_1(q), \dots, X_m(q)\}$  is constant.

However in general this correspondence does not hold globally since, for topological reasons, a distribution of rank  $m$  may not always be generated by  $m$  vector fields on the whole  $M$ . Conversely, the vector fields  $X_1, \dots, X_m$  of a nonholonomic system do not always generate a linear space  $\Delta(q)$  of constant rank. It may even be impossible, again for topological reasons (for instance, on an even dimensional sphere).

A way to conciliate both notions is to generalize the definition of sub-Riemannian structure.

**Definition.** A *generalized sub-Riemannian structure* on  $M$  is a triple  $(E, \sigma, g_R)$  where

- $E$  is a vector bundle over  $M$ ;
- $\sigma : E \rightarrow TM$  is a morphism of vector bundles;
- $g_R$  is a Riemannian metric on  $E$ .

To a generalized sub-Riemannian structure is associated a metric defined, for  $q \in M$  and  $v \in T_qM$ , by

$$g_{SR}(q, v) = \inf\{g(q, u) : u \in E(q), \sigma(u) = v\}.$$

The length of absolutely continuous paths and the generalized sub-Riemannian distance  $d_{SR}$  associated to  $(E, \sigma, g_R)$  are defined as above.

This definition of sub-Riemannian distance actually contains the two notions of distance we have introduced before.

- Take  $E = M \times \mathbb{R}^m$ ,  $\sigma : E \rightarrow TM$ ,  $\sigma(q, u) = \sum_{i=1}^m u_i X_i(q)$  and  $g_R$  the Euclidean metric on  $\mathbb{R}^m$ . The resulting generalized sub-Riemannian distance is the distance associated to the nonholonomic system (1).
- Take  $E = D$ , where  $D$  is a distribution on  $M$ ,  $\sigma : D \hookrightarrow TM$  the inclusion, and  $g_R$  a Riemannian metric on  $D$ . We recover the distance associated to the sub-Riemannian structure  $(D, g_R)$ .

Locally, a generalized sub-Riemannian structure can always be defined by a single finite family  $X_1, \dots, X_m$  of vector fields, and so by a nonholonomic system (without rank condition). It actually appears that it is also true globally (see [ABB12], or [DLPR12] for the fact that a sub-module of  $TM$  is finitely generated): any generalized sub-Riemannian distance may be associated to a nonholonomic system.

In these notes, we will always consider a sub-Riemannian distance  $d$  associated to a nonholonomic system. However, as noticed above, all the results hold for a generalized sub-Riemannian distance.

## 2 Controllability

As previously,  $M$  is a smooth  $n$ -dimensional manifold and  $VF(M)$  denotes the set of smooth vector fields on  $M$ .

### 2.1 The Chow-Rashevsky Theorem

Let  $\dot{q} = \sum_{i=1}^m u_i X_i(q)$  be a nonholonomic system on  $M$ . We define  $\Delta^1$  to be the subset of  $VF(M)$  generated by  $X_1, \dots, X_m$ ,

$$\Delta^1 = \text{span}\{X_1, \dots, X_m\}.$$

For  $s \geq 1$ , define  $\Delta^{s+1} = \Delta^s + [\Delta^1, \Delta^s]$ , where we have set  $[\Delta^1, \Delta^s] = \text{span}\{[X, Y] : X \in \Delta^1, Y \in \Delta^s\}$ . The Lie algebra generated by  $X_1, \dots, X_m$  is defined to be  $\text{Lie}(X_1, \dots, X_m) = \bigcup_{s \geq 1} \Delta^s$ . Due to the Jacobi identity, it is the smallest linear subspace of  $VF(M)$  containing  $X_1, \dots, X_m$  which is left invariant by Lie brackets.

Let us denote by  $I = i_1 \cdots i_k$  a multi-index of  $\{1, \dots, m\}$ , and by  $|I| = k$  the length  $I$ . We set

$$X_I = [X_{i_1}, [\dots, [X_{i_{k-1}}, X_{i_k}] \dots]].$$

With these notations,  $\Delta^s = \text{span}\{X_I : |I| \leq s\}$ .

For  $q \in M$ , we set  $\text{Lie}(X_1, \dots, X_m)(q) = \{X(q) : X \in \text{Lie}(X_1, \dots, X_m)\}$ , and, for  $s \geq 1$ ,  $\Delta^s(q) = \{X(q) : X \in \Delta^s\}$ . Both of them are linear subspaces of  $T_q M$ .

**Definition.** We say that the vector fields  $X_1, \dots, X_m$  satisfy *Chow's Condition* if,  $\forall q \in M$ ,  $\text{Lie}(X_1, \dots, X_m)(q) = T_q M$ .

Equivalently, for any  $q \in M$ , there exists an integer  $r = r(q)$  such that  $\dim \Delta^r(q) = n$ .

This property is also known as Lie algebra rank condition (LARC), and as Hörmander condition (in the context of PDE).

**Lemma 2.1.** *Let  $p$  be a point in  $M$  and  $\mathcal{A}_p$  be the set of points joined to  $p$  by a trajectory of (1). If  $X_1, \dots, X_m$  satisfy Chow's Condition, then  $\mathcal{A}_p$  is a neighbourhood of  $p$ .*

*Proof.* Let us work in a small neighbourhood  $U \subset M$  of  $p$  that we identify with a neighbourhood of 0 in  $\mathbb{R}^n$ .

Let  $\phi_t^i$  be the flow of the vector field  $X_i$ . The curves  $t \mapsto \phi_t^i(q)$  are trajectories and we have

$$\phi_t^i = \text{id} + tX_i + o(t).$$

Define then  $\phi_t^{ij}$  as the commutator of flows, that is

$$\phi_t^{ij} = [\phi_t^i, \phi_t^j] = \phi_{-t}^j \circ \phi_{-t}^i \circ \phi_t^j \circ \phi_t^i.$$

It is a well-known fact that  $\phi_t^{ij} = \text{id} + t^2 X_{ij} + o(t^2)$ .

For a multi-index  $I = iJ$ , we define by induction the local diffeomorphisms  $\phi_t^I = [\phi_t^i, \phi_t^J]$ . Thus  $\phi_t^I(q)$  is the endpoint of a trajectory. Moreover it results from the Campbell-Hausdorff formula that

$$\phi_t^I = \text{id} + t^{|I|} X_I + o(t^{|I|}).$$

To obtain a diffeomorphism whose derivative with respect to the time is exactly  $X_I$ , we set

$$\psi_t^I = \begin{cases} \phi_{t^{1/|I|}}^I & \text{if } t \geq 0, \\ \phi_{-|t|^{1/|I|}}^I & \text{if } t < 0 \text{ and } |I| \text{ is odd,} \\ [\phi_{|t|^{1/|I|}}^J, \phi_{|t|^{1/|I|}}^i] & \text{if } t < 0 \text{ and } |I| \text{ is even.} \end{cases}$$

We have

$$\psi_t^I = \text{id} + t X_I + o(t), \quad (5)$$

and  $\psi_t^I(q)$  is the endpoint of a trajectory.

Let us choose now commutators  $X_{I_1}, \dots, X_{I_n}$  whose values at  $p$  span  $T_p M$ . It is possible since  $X_1, \dots, X_m$  satisfy Chow's Condition. Introduce the map  $\varphi$  defined on a small neighbourhood  $\Omega$  of 0 in  $\mathbb{R}^n$  by

$$\varphi(t_1, \dots, t_n) = \psi_{t_n}^{I_n} \circ \dots \circ \psi_{t_1}^{I_1}(p) \in M.$$

Due to (5), this map is  $C^1$  near 0 and its derivative at 0 is invertible. This implies that  $\varphi$  is a local  $C^1$ -diffeomorphism and so that  $\varphi(\Omega)$  contains a neighbourhood of  $p$ .

Now, for every  $t \in \Omega$ ,  $\varphi(t)$  results from a concatenation of trajectories, the first one issued from  $p$ . It is then the endpoint of a trajectory starting from  $p$ . Therefore  $\varphi(\Omega) \subset \mathcal{A}_p$ , which implies that  $\mathcal{A}_p$  is a neighbourhood of  $p$ .  $\square$

**Theorem 2.2** (Chow-Rashevsky's theorem). *If  $M$  is connected and if  $X_1, \dots, X_m$  satisfy Chow's Condition, then any two points of  $M$  can be joined by a trajectory, and so  $d < \infty$ .*

*Proof.* Let  $p \in M$ . If  $q \in \mathcal{A}_p$ , then  $p \in \mathcal{A}_q$ . As a consequence,  $\mathcal{A}_p = \mathcal{A}_q$  for any  $q \in M$  and the lemma above then implies that  $\mathcal{A}_p$  is an open set. Hence the manifold  $M$  is covered by the union of the sets  $\mathcal{A}_p$  that are disjoint from each other. Since  $M$  is connected, there is only one such open set.  $\square$

*Remark.* This theorem appears also as a consequence of the Orbit Theorem (Sussmann, Stefan): each set  $\mathcal{A}_p$  is a connected immersed submanifold of  $M$  and, at each point  $q \in \mathcal{A}_p$ ,  $\text{Lie}(X_1, \dots, X_m)(q) \subset T_q \mathcal{A}_p$ . Moreover, when the rank of the Lie algebra is constant on  $M$ , both spaces are equal:  $\text{Lie}(X_1, \dots, X_m)(q) = T_q \mathcal{A}_p$ .

Thus, when the Lie algebra generated by  $X_1, \dots, X_m$  has constant rank, Chow's Condition is not restrictive: it is indeed satisfied on each  $\mathcal{A}_p$  by the restriction to the manifold  $\mathcal{A}_p$  of the vector fields  $X_1, \dots, X_m$ .

## 2.2 Topological structure of $(M, d)$

The proof of Lemma 2.1 gives a little bit more than the openness of  $\mathcal{A}_p$ . For  $\varepsilon$  small enough, any  $\phi_t^i(q)$ ,  $0 \leq t \leq \varepsilon$  is a trajectory of length  $\varepsilon$ . Thus  $\varphi(t_1, \dots, t_n)$  is the endpoint of a trajectory of length less than  $N(|t_1|^{1/|I_1|} + \dots + |t_n|^{1/|I_n|})$ , where  $N$  counts the maximal number of concatenations involved in the  $\psi_t^{I_i}$ 's. This gives an upper bound for the distance:

$$d(p, \varphi(t)) \leq N(|t_1|^{1/|I_1|} + \dots + |t_n|^{1/|I_n|}). \quad (6)$$

This kind of estimates of the distance in function of local coordinates plays an important role in sub-Riemannian geometry. However here  $(t_1, \dots, t_n)$  are not local coordinates:  $\varphi$  is only a  $C^1$ -diffeomorphism, not a smooth diffeomorphism.

Let's try to replace  $(t_1, \dots, t_n)$  by local coordinates. Choose local coordinates  $(y_1, \dots, y_n)$  centered at  $p$  such that  $\frac{\partial}{\partial y_i}|_p = X_{I_i}(p)$ . The map  $\varphi^y = y \circ \varphi$  is a  $C^1$ -diffeomorphism between neighbourhoods of 0 in  $\mathbb{R}^n$ , and its differential at 0 is  $d\varphi_0^y = \text{Id}_{\mathbb{R}^n}$ .

Denoting by  $\|\cdot\|_{\mathbb{R}^n}$  the Euclidean norm on  $\mathbb{R}^n$ , we obtain, for  $\|t\|_{\mathbb{R}^n}$  small enough,  $y_i(t) = t_i + o(\|t\|_{\mathbb{R}^n})$ . The inequality (6) becomes

$$d(p, q^y) \leq N' \|y\|_{\mathbb{R}^n}^{1/r},$$

where  $q^y$  denotes the point of coordinates  $y$  and  $r = \max_i |I_i|$ . This inequality allows to compare  $d$  to a Riemannian distance.

Let  $g^R$  be a Riemannian metric on  $M$ , and  $d_R$  the associate Riemannian distance. On a compact neighbourhood of  $p$ , there exists a constant  $c > 0$  such that  $g(X_i, X_i)(q) \leq c^{-1}$ , which implies  $cd_R(p, q) \leq d(p, q)$ . Moreover we have  $d_R(p, q^y) \geq Cst \|y\|_{\mathbb{R}^n}$ . We then obtain a first estimate of the sub-Riemannian distance.

**Theorem 2.3.** *Assume  $X_1, \dots, X_m$  satisfy Chow's Condition. For any Riemannian metric  $g^R$ , we have, for  $q$  close enough to  $p$ ,*

$$cd_R(p, q) \leq d(p, q) \leq Cd_R(p, q)^{1/r}$$

where  $c, C$  are positive constants and  $r$  is an integer such that  $\Delta_p^r = T_p M$ .

*Remark.* If we choose for  $g^R$  a Riemannian metric which is compatible with  $g$ , that is  $g^R|_\Delta = g$ , then by construction  $d_R(p, q) \leq d(p, q)$ .

As a consequence the sub-Riemannian distance  $d$  is  $1/r$ -Hölder with respect to any Riemannian distance, and so continuous.

**Corollary 2.4.** *If  $X_1, \dots, X_m$  satisfy Chow's Condition, then the topology of the metric space  $(M, d)$  is the original topology of  $M$ .*

*Remark.* The converse of Chow's theorem is false in general. Consider for instance the nonholonomic system in  $\mathbb{R}^3$  defined by  $X_1 = \partial_x$ ,  $X_2 = \partial_y + f(x)\partial_z$  where  $f(x) = e^{-1/x^2}$  for positive  $x$  and  $f(x) = 0$  otherwise. The associated sub-Riemannian distance is finite whence  $X_1, \dots, X_m$  do not satisfy Chow's Condition. Moreover the topology of the metric space is different from the canonical topology of  $\mathbb{R}^3$ .

However, for an analytic nonholonomic system (that is when  $M$  and the vector fields  $X_1, \dots, X_m$  are in the analytic category), Chow's Condition is equivalent to the finiteness of the distance.

### 3 First-order approximations

Consider a nonholonomic system  $\dot{q} = \sum_{i=1}^m u_i X_i(q)$  on a manifold  $M$  and the induced sub-Riemannian distance  $d$ . The infinitesimal behaviour of a nonholonomic system should be captured by an approximation to the first-order. However the latter notion has to be carefully defined. Take for instance the usual notion of order at a point  $p$ , that is the one induced by the differential structure of the manifold  $M$ . The first-order approximation at  $p$  of the  $X_1, \dots, X_m$  is the family of constant vector fields  $X_1(p), \dots, X_m(p)$  (defined in coordinates on a neighbourhood of  $p$ ). Being constant, these vector fields can of course not satisfy Chow's Condition, and so can not reflect the local properties of the nonholonomic system, except in the trivial case where  $\Delta(p) = T_p M$ .

In this section we will then revisit the notion of first-order approximation and construct the basis of an infinitesimal calculus adapted to the nonholonomic systems. The basic concept is the one of nonholonomic order of a function at a point. We will then see that approximations to the first-order appear as nilpotent approximations,  $X_1, \dots, X_m$  being approximated by vector fields that generate a nilpotent Lie algebra.

The whole section is concerned with local objects. We then fix a point  $p \in M$  and an open neighbourhood  $U$  of  $p$  that we identify with a neighbourhood



of 0 in  $\mathbb{R}^n$  in some local coordinates. We also assume that  $X_1, \dots, X_m$  satisfy Chow's Condition.

### 3.1 Nonholonomic orders

**Definition.** Let  $f : M \rightarrow \mathbb{R}$  be a continuous function. The *nonholonomic order of  $f$  at  $p$* , denoted by  $\text{ord}_p(f)$ , is the real number defined as:

$$\text{ord}_p(f) = \sup \{s \in \mathbb{R} : f(q) = O(d(p, q)^s)\}.$$

This order is always nonnegative. We have  $\text{ord}_p(f) = 0$  if  $f(p) \neq 0$ , and  $\text{ord}_p(f) = +\infty$  if  $f(p) \equiv 0$ .

In the Euclidean case, that is when  $M = \mathbb{R}^n$ ,  $m = n$ , and  $X_i = \partial_{x_i}$ , nonholonomic orders coincide with the usual vanishing orders of the analysis. At  $x = 0$ , it is the smallest degree of the monomials which appears with a nonzero coefficient in the Taylor series

$$f(x) \sim \sum c_\alpha x_1^{\alpha_1} \dots x_n^{\alpha_n}$$

of  $f$  at 0. We will see now that there exists an analogous characterization of nonholonomic orders.

Let  $C^\infty(p)$  denote the set of germs of smooth functions at  $p$ . For a function  $f \in C^\infty(p)$ , we call *nonholonomic derivatives of order 1 of  $f$*  the Lie derivatives  $X_1 f, \dots, X_m f$ . We call further  $X_i(X_j f)$ ,  $X_i(X_j(X_k f))$ , ... the *nonholonomic derivatives of  $f$  of order 2, 3, ...*. The nonholonomic derivative of order 0 of  $f$  at  $p$  is  $f(p)$ .

**Proposition 3.1.** *Let  $f \in C^\infty(p)$ . Then  $\text{ord}_p(f)$  is the biggest integer  $k$  such that all nonholonomic derivatives of  $f$  of order smaller than  $k$  vanish at  $p$ . Moreover,*

$$f(q) = O(d(p, q)^{\text{ord}_p(f)}).$$

*Proof.* The proposition results from the two following facts:

- (i) if  $\ell$  is an integer such that  $\ell < \text{ord}_p(f)$ , then all nonholonomic derivatives of  $f$  of order  $\leq \ell$  vanish at  $p$ ;
- (ii) if  $\ell$  is an integer such that all nonholonomic derivatives of  $f$  of order  $\leq \ell$  vanish at  $p$ , then  $f(q) = O(d(p, q)^{\ell+1})$ .

Let us first prove point (i). Let  $\ell$  be an integer such that  $\ell < \text{ord}_p(f)$ . We write a nonholonomic derivative of  $f$  of order  $k \leq \ell$  as

$$(X_{i_1} \dots X_{i_k} f)(p) = \frac{\partial^k}{\partial t_1 \dots \partial t_k} f(e^{t_k X_{i_k}} \circ \dots \circ e^{t_1 X_{i_1}}(p)) \Big|_{t=0}.$$

The point  $q = e^{t_k X_{i_k}} \circ \dots \circ e^{t_1 X_{i_1}}(p)$  is the endpoint of a trajectory of length  $|t_1| + \dots + |t_n|$ . Therefore,  $d(p, q) \leq |t_1| + \dots + |t_n|$ .

Since  $k \leq \ell < \text{ord}_p(f)$ , there exists a real number  $s > 0$  such that  $f(q) = O((|t_1| + \dots + |t_n|)^{k+s})$ . This implies that

$$(X_{i_1} \dots X_{i_k} f)(p) = \frac{\partial^k}{\partial t_1 \dots \partial t_k} f(q) \Big|_{t=0} = 0.$$

Thus point (i) is proved.

The proof of point (ii) goes by induction on  $\ell$ .

For  $\ell = 0$ , assume that all nonholonomic derivatives of  $f$  of order  $\leq 0$  vanish at  $p$ , that is  $f(p) = 0$ . Choose any Riemannian metric on  $M$  and denote by  $d_R$  the associated Riemannian distance on  $M$ . There holds  $f(q) \leq Cst d_R(p, q)$  near  $p$ . By Theorem 2.3, this implies  $f(q) \leq Cst d(p, q)$ , and so property (ii) for  $\ell = 0$ .

Assume that, for a given  $\ell \geq 0$ , (ii) holds for any function  $f$  (induction hypothesis) and take a function  $f$  such that all its nonholonomic derivatives of order  $< \ell + 1$  vanish at  $p$ .

Observe that, for  $i = 1, \dots, m$ , all the nonholonomic derivatives of  $X_i f$  of order  $< \ell$  vanish at  $p$ . Indeed,  $X_{i_1} \dots X_{i_k}(X_i f) = X_{i_1} \dots X_{i_k} X_i f$ . Applying the induction hypothesis to  $X_i f$  yields  $X_i f(q) = O(d(p, q)^\ell)$ . In other terms, there exist positive constants  $C_1, \dots, C_m$  such that, for  $q$  close enough to  $p$ ,

$$X_i f(q) \leq C_i d(p, q)^\ell.$$

Fix now a point  $q$  near  $p$ . There exists a minimizing curve  $\gamma(\cdot)$  joining  $p$  to  $q$ , which can be assumed of velocity one. This means that  $\gamma$  satisfies

$$\dot{\gamma}(t) = \sum_{i=1}^m u_i(t) X_i(\gamma(t)) \quad \text{for a.e. } t \in [0, T], \quad \gamma(0) = p, \quad \gamma(T) = q,$$

with  $\sum_i u_i^2(t) = 1$  a.e. and  $d(p, q) = \text{length}(\gamma) = T$ . Actually every sub-arc of  $\gamma$  is also clearly minimizing, so  $d(p, \gamma(t)) = t$  for any  $t \in [0, T]$ .

To estimate  $f(q) = f(\gamma(T))$ , we compute the derivative of  $f(\gamma(t))$  with respect to  $t$ :

$$\begin{aligned} \frac{d}{dt} f(\gamma(t)) &= \sum_{i=1}^m u_i(t) X_i f(\gamma(t)), \\ \Rightarrow \left| \frac{d}{dt} f(\gamma(t)) \right| &\leq \sum_{i=1}^m |u_i(t)| C_i d(p, \gamma(t))^\ell \leq C t^\ell, \end{aligned}$$

where  $C = C_1 + \dots + C_m$ . Integrating this inequality between 0 and  $t$  yields

$$|f(\gamma(t))| \leq |f(p)| + \frac{C}{\ell + 1} t^{\ell+1}.$$

We have  $f(p) = 0$  since it is the nonholonomic derivative of  $f$  of order 0. Finally, at  $t = T = d(p, q)$ , we obtain

$$|f(q)| \leq \frac{C}{\ell + 1} T^{\ell+1},$$

which concludes the proof of (ii).  $\square$

As a consequence, the nonholonomic order of a smooth (germ of) function is given by the formula:

$$\text{ord}_p(f) = \min \{s \in \mathbb{N} : \exists i_1, \dots, i_s \in \{1, \dots, m\} \text{ s.t. } (X_{i_1} \dots X_{i_s} f)(p) \neq 0\},$$

where as usual we adopt the convention that  $\min \emptyset = +\infty$ .

It is clear now that any function in  $C^\infty(p)$  vanishing at  $p$  is of order  $\geq 1$ . Moreover, the following basic computation rules are satisfied: for every functions  $f, g$  in  $C^\infty(p)$  and every real number  $\lambda \neq 0$ ,

$$\begin{aligned} \text{ord}_p(fg) &\geq \text{ord}_p(f) + \text{ord}_p(g), \\ \text{ord}_p(\lambda f) &= \text{ord}_p(f), \\ \text{ord}_p(f + g) &\geq \min(\text{ord}_p(f), \text{ord}_p(g)). \end{aligned}$$

Notice that the first inequality is actually an equality. However the proof of this fact requires an additional result (see Proposition 3.2).

The notion of nonholonomic order extends to vector fields. Let  $VF(p)$  denote the set of germs of vector fields at  $p$ .

**Definition.** Let  $X \in VF(p)$ . The *nonholonomic order of  $X$  at  $p$* , denoted by  $\text{ord}_p(X)$ , is the relative integer defined as:

$$\text{ord}_p(X) = \sup \{\sigma \in \mathbb{R} : \text{ord}_p(Xf) \geq \sigma + \text{ord}_p(f), \quad \forall f \in C^\infty(p)\}.$$

The order of a differential operator is defined in exactly the same way.

The fact that  $\text{ord}_p(X) \in \mathbb{Z}$  arises from the fact that the order of a smooth function is an integer. Note also that the null vector field is of infinite order,  $\text{ord}_p(0) = +\infty$ .

The order of a function coincides with its order as a differential operator acting by multiplication. We have then the following properties, for every vector fields  $X, Y$  and every function  $f \in C^\infty(p)$ :

$$\begin{aligned} \text{ord}_p([X, Y]) &\geq \text{ord}_p(X) + \text{ord}_p(Y), \\ \text{ord}_p(fX) &\geq \text{ord}_p(f) + \text{ord}_p(X), \\ \text{ord}_p(X) &\leq \text{ord}_p(Xf) - \text{ord}_p(f), \\ \text{ord}_p(X + Y) &\geq \min(\text{ord}_p(X), \text{ord}_p(Y)). \end{aligned}$$

As already noticed for functions, the second inequality is in fact an equality. This is not the case for the first inequality (take for instance commuting nonzero vector fields).

As a consequence,  $X_1, \dots, X_m$  are of order  $\geq -1$ ,  $[X_i, X_j]$  of order  $\geq -2$ , and more generally, if  $X$  belongs to the set  $\text{Lie}^k(X_1, \dots, X_m)$ , then it is of order  $\geq -k$ .

Note that in the Euclidean case (that is, when  $M = \mathbb{R}^n$ ,  $m = n$ , and  $X_i = \partial_{x_i}$ ), the nonholonomic order of a constant differential operator is the negative of its usual order. For instance  $\partial_{x_i}$  is of nonholonomic order  $-1$ . Actually, every vector field that do not vanish at  $p$  is of nonholonomic order  $-1$ .

*Example 3.1* (Heisenberg case). Consider the following vector fields on  $\mathbb{R}^3$ :

$$X_1 = \partial_x - \frac{y}{2}\partial_z \quad \text{and} \quad X_2 = \partial_y + \frac{x}{2}\partial_z.$$

At 0, the coordinates  $x$  and  $y$  are of order 1 and  $z$  is of order 2 since  $X_1x(0) = X_2y(0) = 1$ ,  $X_1z(0) = X_2z(0) = 0$ , and  $X_1X_2z(0) = 1/2$ . These relations also imply  $\text{ord}_0(X_1) = \text{ord}_0(X_2) = -1$ . Finally, the bracket  $[X_1, X_2] = \partial_z$  is of order  $-2$  since  $[X_1, X_2]z = 1$ .

We are now in a position to precise the meaning of first-order approximation.

**Definition.** A family of  $m$  vector fields  $\widehat{X}_1, \dots, \widehat{X}_m$  defined near  $p$  is called a *first-order approximation of  $X_1, \dots, X_m$  at  $p$*  if the vector fields  $X_i - \widehat{X}_i$ ,  $i = 1, \dots, m$ , are of order  $\geq 0$  at  $p$ .

In particular the order at  $p$  defined by the vector fields  $\widehat{X}_1, \dots, \widehat{X}_m$  coincides with the one defined by  $X_1, \dots, X_m$ : for any  $f \in C^\infty(p)$  of order greater than  $k - 1$ ,

$$(X_{i_1} \dots X_{i_s} f)(q) = (\widehat{X}_{i_1} \dots \widehat{X}_{i_s} f)(q) + O(d(p, q)^{\text{ord}_p(f) - s + 1}).$$

To go further in the characterization of orders and approximations, we need suitable systems of coordinates.

### 3.2 Privileged coordinates

We have introduced in subsection 2.1 the sets of vector fields  $\Delta^s$ , defined as  $\Delta^s = \text{span}\{X_I : |I| \leq s\}$ . Since  $X_1, \dots, X_m$  satisfy Chow's Condition, the values of these sets at  $p$  form a flag of subspaces of  $T_pM$ :

$$\Delta^1(p) \subset \Delta^2(p) \subset \dots \subset \Delta^{r-1}(p) \subsetneq \Delta^r(p) = T_pM, \quad (7)$$

where  $r = r(p)$  is called the *degree of nonholonomy at  $p$* .

Set  $n_i(p) = \dim \Delta^i(p)$ . The integer list  $(n_1(p), \dots, n_r(p))$  is called the *growth vector at  $p$* . The first integer  $n_1(p)$  is the rank of the family  $(X_1(p), \dots, X_m(p))$  and the last one  $n_r(p) = n$  is the dimension of the manifold  $M$ .

Let  $s \geq 1$ . By abuse of notations, we continue to write  $\Delta^s$  for the map  $q \mapsto \Delta^s(q)$ . This map  $\Delta^s$  is a distribution if and only if  $n_s(q)$  is constant on  $M$ . We then distinguish two kind of points.

**Definition.** The point  $p$  is a *regular point* if the growth vector is constant in a neighbourhood of  $p$ . Otherwise,  $p$  is a *singular point*.

Thus, near a regular point, all maps  $\Delta^s$  are locally distributions.

The structure of the flag (7) may also be described by another sequence of integers. We define the *weights at  $p$* ,  $w_i = w_i(p)$ ,  $i = 1, \dots, n$ , by setting  $w_j = s$  if  $n_{s-1}(p) < j \leq n_s(p)$  (we have set  $n_0 = 0$ ). In other terms, we have

$$w_1 = \dots = w_{n_1} = 1, \quad w_{n_1+1} = \dots = w_{n_2} = 2, \quad \dots \\ w_{n_{r-1}+1} = \dots = w_n = r.$$

The weights at  $p$  form an increasing sequence  $w_1(p) \leq \dots \leq w_n(p)$  which is constant near  $p$  if and only if  $p$  is a regular point.

*Example 3.2.* The Heisenberg case in  $\mathbb{R}^3$  given in example 3.1 has a growth vector which is equal to  $(2, 3)$  at every point. All points of  $\mathbb{R}^3$  are then regular. The weights at any point are  $w_1 = w_2 = 1$ ,  $w_3 = 2$ .

*Example 3.3* (Martinet case). Consider the following vector fields on  $\mathbb{R}^3$ :

$$X_1 = \partial_x \quad \text{and} \quad X_2 = \partial_y + \frac{x^2}{2} \partial_z.$$

The growth vector is equal to  $(2, 2, 3)$  on the plane  $\{y = 0\}$ , and to  $(2, 3)$  elsewhere. As a consequence, the set of singular points is the plane  $\{y = 0\}$ . The weights are  $w_1 = w_2 = 1$ ,  $w_3 = 2$  at regular points, and  $w_1 = w_2 = 1$ ,  $w_3 = 3$  at singular ones.

*Example 3.4.* Consider the vector fields on  $\mathbb{R}^3$

$$X_1 = \partial_x \quad \text{and} \quad X_2 = \partial_y + f(x)\partial_z,$$

where  $f$  is a smooth function on  $\mathbb{R}$  which admits every positive integer  $n \in \mathbb{N}$  as a zero with multiplicity  $n$  (such a function exists and can even be chosen in the analytic class thanks to the Weierstrass factorization theorem). Every point  $(n, y, z)$  is then singular and the weights at these points are  $w_1 = w_2 = 1$ ,  $w_3 = n + 1$ . As a consequence the degree of nonholonomy  $w_3$  is unbounded on  $\mathbb{R}^3$ .

Let us give some basic properties of the growth vector and of the weights.

- At a regular point, the growth vector is a strictly increasing sequence:  $n_1(p) < \dots < n_r(p)$ . Indeed, if  $n_s(q) = n_{s+1}(q)$  in a neighbourhood of  $p$ , then  $\Delta^s$  is locally an involutive distribution and so  $s = r$ . As a consequence, at a regular point  $p$ , the jump between two successive weights is never greater than 1,  $w_{i+1} - w_i \leq 1$ , and there holds  $r(p) \leq n - m + 1$ .
- For every  $s$ , the map  $q \mapsto n_s(q)$  defines a lower semi-continuous function from  $M$  to  $\mathbb{N}$ . Therefore the regular points form an open and dense subset of  $M$ .
- For every  $i = 1, \dots, n$ , the weight  $w_i(\cdot)$  is an upper semi-continuous function. It is in particular the case for the degree of nonholonomy  $r(\cdot) = w_n(\cdot)$ , that is  $r(q) \leq r(p)$  for  $q$  near  $p$ . As a consequence  $r(\cdot)$  is bounded on any compact subset of  $M$ .
- The degree of nonholonomy may be unbounded on  $M$  (see example 3.4 above). Thus the finiteness of a sub-Riemannian distance is a non decidable problem: the computation of an infinite number of brackets may be needed to decide if Chow's Condition is satisfied.

However in the case (important in practice) of polynomial vector fields on  $\mathbb{R}^n$ , it can be shown that the degree of nonholonomy is bounded by a universal function of the degree  $d$  of the polynomials (see [Gab95, GJR98]):

$$r(x) \leq 2^{3n^2} n^{2n} d^{2n}.$$

The meaning of the sequence of weights is best understood in terms of basis of  $T_p M$ . Choose first vector fields  $Y_1, \dots, Y_{n_1}$  in  $\Delta^1$  which values at  $p$  form a basis of  $\Delta^1(p)$ . Choose then vector fields  $Y_{n_1+1}, \dots, Y_{n_2}$  in  $\Delta^2$  such that the values  $Y_1(p), \dots, Y_{n_2}(p)$  form a basis of  $\Delta^2(p)$ . For each  $s$ , choose

$Y_{n_{s-1}+1}, \dots, Y_{n_s}$  in  $\Delta^s$  such that  $Y_1(p), \dots, Y_{n_s}(p)$  form a basis of  $\Delta^s(p)$ . We obtain in this way a family of vector fields  $Y_1, \dots, Y_n$  such that

$$\begin{cases} Y_1(p), \dots, Y_n(p) \text{ is a basis of } T_p M, \\ Y_i \in \Delta^{w_i}, \quad i = 1, \dots, n. \end{cases} \quad (8)$$

A family of  $n$  vector fields satisfying (8) is called an *adapted frame at  $p$* . The word ‘‘adapted’’ means ‘‘adapted to the flag (7)’’, since the values at  $p$  of an adapted frame contain a basis  $Y_1(p), \dots, Y_{n_s}(p)$  of each subspace  $\Delta^s(p)$  of the flag. By continuity, at a point  $q$  close enough to  $p$ , the values of  $Y_1, \dots, Y_n$  still form a basis of  $T_q M$ . However, if  $p$  is singular, this basis may be not adapted to the flag (7) at  $q$ .

Let us exhibit now the relation between weights and orders. We write first the tangent space as a direct sum:

$$T_p M = \Delta^1(p) \oplus \Delta^2(p)/\Delta^1(p) \oplus \dots \oplus \Delta^r(p)/\Delta^{r-1}(p),$$

where  $\Delta^s(p)/\Delta^{s-1}(p)$  denotes a supplementary of  $\Delta^{s-1}(p)$  in  $\Delta^s(p)$ , and take a local system of coordinates  $(y_1, \dots, y_n)$ . The dimension of each space  $\Delta^s(p)/\Delta^{s-1}(p)$  is equal to  $n_s - n_{s-1}$ , so we can assume that, up to a reordering, we have  $dy_j(\Delta^s(p)/\Delta^{s-1}(p)) \neq 0$  for  $n_{s-1} < j \leq n_s$ .

Take an integer  $j$  such that  $0 < j \leq n_1$ . From the above assumption, there holds  $dy_j(\Delta^1(p)) \neq 0$ . There exists then  $X_i$  such that  $dy_j(X_i(p)) \neq 0$ . Since  $dy_j(X_i) = X_i y_j$  is a first-order nonholonomic derivative of  $y_j$ , we have  $\text{ord}_p(y_j) \leq 1 = w_j$ .

Take now an integer  $j$  such that  $n_{s-1} < j \leq n_s$  for  $s > 1$ , that is  $w_j = s$ . Since  $dy_j(\Delta^s(p)/\Delta^{s-1}(p)) \neq 0$ , there exists a vector field  $Y$  in  $\Delta^s$  such that  $dy_j(Y(p)) = (Y y_j)(p) \neq 0$ . By definition of  $\Delta^s$ , the Lie derivative  $Y y_j$  is a linear combination of nonholonomic derivatives of  $y_j$  of order not greater than  $s$ . One of them must be nonzero and so  $\text{ord}_p(y_j) \leq s = w_j$ .

To sum up, for any system of local coordinates  $(y_1, \dots, y_n)$ , we have, up to a reordering,  $\text{ord}_p(y_j) \leq w_j$  (or, without reordering,  $\sum_{i=1}^n \text{ord}_p(y_i) \leq \sum_{i=1}^n w_i$ ). The coordinates with the maximal possible order will play an important role in the sequel.

**Definition.** A *system of privileged coordinates at  $p$*  is a system of local coordinates  $(z_1, \dots, z_n)$  such that  $\text{ord}_p(z_j) = w_j$  for  $j = 1, \dots, n$ .

Notice that privileged coordinates  $(z_1, \dots, z_n)$  satisfy

$$dz_i(\Delta^{w_i}(p)) \neq 0, \quad dz_i(\Delta^{w_i-1}(p)) = 0, \quad i = 1, \dots, n, \quad (9)$$

or, equivalently,  $\partial_{z_i}|_p$  belongs to  $\Delta^{w_i}(p)$  but not to  $\Delta^{w_i-1}(p)$ . Local coordinates satisfying (9) are called *linearly adapted coordinates* (‘‘adapted’’ because

the differentials at  $p$  of the coordinates form a basis of  $T_p^*M$  dual to the values of an adapted frame). Thus privileged coordinates are always linearly adapted coordinates. The converse is false, as shown in the example below.

*Example 3.5.* Take  $X_1 = \partial_x$ ,  $X_2 = \partial_y + (x^2 + y)\partial_z$  in  $\mathbb{R}^3$ . The weights at 0 are  $(1, 1, 3)$  and  $(x, y, z)$  are adapted at 0. But they are not privileged: indeed, the coordinate  $z$  is of order 2 at 0 since  $(X_2 X_2 z)(0) = 1$ .

*Remark.* As it is proposed by Kupka [Kup96], one can define *privileged functions at  $p$*  as the smooth functions  $f$  on  $U$  such that

$$\text{ord}_p(f) = \min\{s \in \mathbb{N} : df(\Delta^s(p)) \neq 0\}.$$

It results from the discussion above that some local coordinates  $(z_1, \dots, z_n)$  are a system of privileged coordinates at  $p$  if and only if each  $z_i$  is a privileged function at  $p$ .

Let us now show how to compute orders using privileged coordinates.

We fix a system of privileged coordinates  $(z_1, \dots, z_n)$  at  $p$ . Given a sequence of integers  $\alpha = (\alpha_1, \dots, \alpha_n)$ , we define the weighted degree of the monomial  $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$  as  $w(\alpha) = w_1 \alpha_1 + \cdots + w_n \alpha_n$  and the weighted degree of the monomial vector field  $z^\alpha \partial_{z_j}$  as  $w(\alpha) - w_j$ . The weighted degrees allow to compute the orders of functions and vector fields in a purely algebraic way.

**Proposition 3.2.** *For a smooth function  $f$  with a Taylor expansion*

$$f(z) \sim \sum_{\alpha} c_{\alpha} z^{\alpha},$$

*the order of  $f$  is the least weighted degree of a monomial appearing in the Taylor series with a nonzero coefficient.*

*For a vector field  $X$  with a Taylor expansion*

$$X(z) \sim \sum_{\alpha, j} a_{\alpha, j} z^{\alpha} \partial_{z_j},$$

*the order of  $X$  is the least weighted degree of a monomial vector field appearing in the Taylor series with a nonzero coefficient.*

In other terms, when using privileged coordinates, the notion of nonholonomic order amounts to the usual notion of vanishing order at some point, only assigning weights to the variables.



*Proof.* For  $i = 1, \dots, n$ , we have  $\partial_{z_i}|_p \in \Delta^{w_i}(p)$ . We can then find  $n$  vector fields  $Y_1, \dots, Y_n$  which form an adapted frame at  $p$  and such that  $Y_1(p) = \partial_{z_1}|_p, \dots, Y_n(p) = \partial_{z_n}|_p$ . For each  $i$ , the vector field  $Y_i$  is of order  $\geq -w_i$  at  $p$  since it belongs to  $\Delta^{w_i}$ . Moreover we have  $(Y_i z_i)(p) = 1$  and  $\text{ord}_p(z_i) = w_i$ . Thus  $\text{ord}_p(Y_i) = -w_i$ .

Take a sequence of integers  $\alpha = (\alpha_1, \dots, \alpha_n)$ . The monomial  $z^\alpha$  is of order  $\geq w(\alpha)$  at  $p$  and the differential operator  $Y^\alpha = Y_1^{\alpha_1} \dots Y_n^{\alpha_n}$  is of order  $\geq -w(\alpha)$ . Noticing that  $(Y_i z_j)(p) = 0$  if  $j \neq i$ , we prove easily that  $(Y^\alpha z^\alpha)(p) = \frac{1}{\alpha_1! \dots \alpha_n!} \neq 0$ , which implies  $\text{ord}_p(z^\alpha) = w(\alpha)$ .

In the same way, we obtain that, if  $z^\alpha, z^\beta$  are two different monomials and  $\lambda, \mu$  two nonzero real numbers, then  $\text{ord}_p(\lambda z^\alpha + \mu z^\beta) = \min(w(\alpha), w(\beta))$ . Thus the order of a series is the least weighted degree of monomials actually appearing in it. This shows the result on order of functions.

As a consequence, for any smooth function  $f$ , the order at  $p$  of  $\partial_{z_i} f$  is  $\geq \text{ord}_p(f) - w_i$ . Since moreover  $\partial_{z_i} z_i = 1$ , we obtain that  $\text{ord}_p(\partial_{z_i})$  is equal to  $-w_i$ . The result on the order of vector fields follows.  $\square$

A notion of homogeneity is also naturally associated to privileged coordinates  $(z_1, \dots, z_n)$ . We define first a one-parameter group of dilations

$$\delta_t : (z_1, \dots, z_n) \mapsto (t^{w_1} z_1, \dots, t^{w_n} z_n), \quad t \in \mathbb{R}.$$

Each dilation  $\delta_t$  is a map from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ . For  $t$  small enough, it can be extended to a map from  $U$  to  $U$  (the privileged coordinates  $z$  are assumed to be defined on the whole  $U$ ): just define  $\delta_t(q)$  as the point in  $U$  with coordinates  $\delta_t(z(q))$ , where  $z(q)$  are the coordinates of  $q$ . A dilation  $\delta_t$  acts also on functions and vector fields by pull-back:  $\delta_t^* f = f \circ \delta_t$  and  $\delta_t^* X$  is the vector field such that  $(\delta_t^* X)(\delta_t^* f) = \delta_t^*(Xf)$ .

**Definition.** A function  $f$  is *homogeneous of degree  $s$*  if  $\delta_t^* f = t^s f$ . A vector field  $X$  is *homogeneous of degree  $\sigma$*  if  $\delta_t^* X = t^\sigma X$ .

For a smooth function (resp. for a vector field), this is the same as being a finite sum of monomials (resp. monomial vector fields) of weighted degree  $s$ . As a consequence, if a function  $f$  is homogeneous of degree  $s$ , then it is of order  $s$  at  $p$ .

A typical degree 1 homogeneous function is the so-called *pseudo-norm*, defined as:

$$z \mapsto \|z\| = |z_1|^{1/w_1} + \dots + |z_n|^{1/w_n}.$$

When composed with the coordinates function, it is a (non smooth) function of order 1, that is,

$$\|z(q)\| = O(d(p, q)).$$

Actually, it results from Proposition 3.2 that the order of a function  $f \in C^\infty(p)$  is the least integer  $s$  such that  $f(q) = O(\|z(q)\|^s)$ .

**Examples of privileged coordinates** Of course all the results above on algebraic computation of orders hold only if privileged coordinates do exist. Two types of privileged coordinates are commonly used in the literature.

**a. Exponential coordinates.** Choose an adapted frame  $Y_1, \dots, Y_n$  at  $p$ . The inverse of the local diffeomorphism

$$(z_1, \dots, z_n) \mapsto e^{z_1 Y_1 + \dots + z_n Y_n}(p)$$

defines a system of local privileged coordinates at  $p$ , called *canonical coordinates of the first kind*. These coordinates are mainly used in the context of hypoelliptic operator and for nilpotent Lie groups with right (or left) invariant sub-Riemannian structure. The fact that these coordinates are privileged is proved – in different terms – in [RS76].

The inverse of the local diffeomorphism

$$(z_1, \dots, z_n) \mapsto e^{z_n Y_n} \circ \dots \circ e^{z_1 Y_1}(p)$$

also defines privileged coordinates at  $p$ , called *canonical coordinates of the second kind*. They are easier to work with than the one of the first kind. For instance, in these coordinates, the vector field  $Y_n$  read as  $\partial_{z_n}$ . One can also exchange the order of the flows in the definition to obtain any of the  $Y_i$  as  $\partial_{z_i}$ . The fact that these coordinates are privileged is proved in [Her91] (see also [Mon02]).

*Exercise 1.* Prove that the diffeomorphism

$$(z_1, \dots, z_n) \mapsto e^{z_n Y_n + \dots + z_{s+1} Y_{s+1}} \circ e^{z_s Y_s} \dots \circ e^{z_1 Y_1}(p)$$

induces privileged coordinates. Show in fact that any “mix” between first and second kind canonical coordinates are privileged coordinates.

**b. Algebraic algorithm.** There exist also effective constructions of privileged coordinates (the construction of exponential coordinates is not effective in general since it requires to integrate flows). We present here Bellaïche’s algorithm (see also [Ste86, AS87]).

1. Choose an adapted frame  $Y_1, \dots, Y_n$  at  $p$ .
2. Choose coordinates  $(y_1, \dots, y_n)$  centered at  $p$  such that  $\partial_{y_i}|_p = Y_i(p)$ .

3. Build privileged coordinates  $z_1, \dots, z_n$  by the iterative formula: for  $j = 1, \dots, n$ ,

$$z_j = y_j + \sum_{k=2}^{w_j-1} h_k(y_1, \dots, y_{j-1}),$$

where, for  $k = 2, \dots, w_j - 1$ ,

$$h_k(y_1, \dots, y_{j-1}) = - \sum_{\substack{|\alpha|=k \\ w(\alpha) < w_j}} Y_1^{\alpha_1} \dots Y_{j-1}^{\alpha_{j-1}} \left( y_j + \sum_{q=2}^{k-1} h_q(y) \right) (p) \frac{y_1^{\alpha_1}}{\alpha_1!} \dots \frac{y_{j-1}^{\alpha_{j-1}}}{\alpha_{j-1}!}$$

where  $|\alpha| = \alpha_1 + \dots + \alpha_n$ .

The coordinates  $y_1, \dots, y_n$  are linearly adapted coordinates. Starting from any system of coordinates, they can be obtained by an affine change of coordinates. Privileged coordinates are obtained from linearly adapted coordinates by expressions of the form

$$\begin{aligned} z_1 &= y_1, \\ z_2 &= y_2 + \text{pol}(y_1), \\ &\vdots \\ z_n &= y_n + \text{pol}(y_1, \dots, y_{n-1}), \end{aligned}$$

where each pol is a polynomial function without constant nor linear terms. Note that the inverse change of coordinates is also of triangular form, which makes the computation easier.

To prove that these coordinates are actually privileged, the key result is the following lemma.

**Lemma 3.3.** *A function  $f$  is of order  $\geq s$  at  $p$  if and only if*

$$(Y_1^{\alpha_1} \dots Y_n^{\alpha_n} f)(p) = 0$$

for all  $\alpha$  such that  $w(\alpha) < s$ .

*Remark.* This lemma seems to be an easy consequence of Proposition 3.2 and of its proof. However, in the latter proposition, existence of privileged coordinates is assumed, whence here the aim is to prove this existence.

Roughly speaking, the idea to obtain  $z_j$  from  $y_j$  is the following. For each  $\alpha$  with  $w(\alpha) < w_j$  (and so  $\alpha_j = \dots = \alpha_n = 0$ ), compute  $(Y_1^{\alpha_1} \dots Y_{j-1}^{\alpha_{j-1}} y_j)(p)$ . If it is nonzero, then replace  $y_j$  by  $y_j - (Y_1^{\alpha_1} \dots Y_{j-1}^{\alpha_{j-1}} y_j)(p) \frac{y_1^{\alpha_1}}{\alpha_1!} \dots \frac{y_{j-1}^{\alpha_{j-1}}}{\alpha_{j-1}!}$ . With the new value of  $y_j$ ,  $(Y_1^{\alpha_1} \dots Y_{j-1}^{\alpha_{j-1}} y_j)(p) = 0$ .

### 3.3 Nilpotent approximation

Fix a system of privileged coordinates at  $p$ . We already know that each vector field  $X_i$  is of order  $\geq -1$ . Moreover, for at least one coordinate  $z_j$  among  $z_1, \dots, z_m$ , the derivative  $(X_i z_j)(p)$  is nonzero (since  $dz_j(\Delta_p^1) \neq 0$ ). This implies that all  $X_i$ 's are of order  $-1$ .

In  $z$  coordinates,  $X_i$  has a Taylor expansion

$$X_i(z) \sim \sum_{\alpha, j} a_{\alpha, j} z^\alpha \partial_{z_j},$$

where  $w(\alpha) \geq w_j - 1$  if  $a_{\alpha, j} \neq 0$ . Grouping together the monomial vector fields of same weighted degree, we express  $X_i$  as a series

$$X_i = X_i^{(-1)} + X_i^{(0)} + X_i^{(1)} + \dots$$

where  $X_i^{(s)}$  is a homogeneous vector field of degree  $s$ .

**Proposition 3.4.** *Set  $\widehat{X}_i = X_i^{(-1)}$ ,  $i = 1, \dots, m$ . The family of vector fields  $\widehat{X}_1, \dots, \widehat{X}_m$  is a first-order approximation of  $X_1, \dots, X_m$  at  $p$  and generate a nilpotent Lie algebra of step  $r = w_n$ .*

*Proof.* The fact that the vector fields  $\widehat{X}_1, \dots, \widehat{X}_m$  form a first-order approximation of  $X_1, \dots, X_m$  results from their construction.

Note further that any homogeneous vector field of degree smaller than  $-w_n$  is zero (clear in privileged coordinates). Moreover, if  $X$  and  $Y$  are homogeneous of degree respectively  $k$  and  $l$ , then the bracket  $[X, Y]$  is homogeneous of degree  $k + l$  because  $\delta_t^*[X, Y] = [\delta_t^*X, \delta_t^*Y] = t^{k+l}[X, Y]$ .

Hence, every iterated bracket of the vector fields  $\widehat{X}_1, \dots, \widehat{X}_m$  of length  $k$  (that is, containing  $k$  of these vector field) is homogeneous of degree  $-k$  and is zero if  $k > w_n$ .  $\square$

**Definition.** The family  $(\widehat{X}_1, \dots, \widehat{X}_m)$  is called the *(homogeneous) nilpotent approximation* of  $(X_1, \dots, X_m)$  at  $p$  associated to the coordinates  $z$ .

*Example 3.6.* Consider  $X_1 = \cos \theta \partial_x + \sin \theta \partial_y$ ,  $X_2 = \partial_\theta$  in  $\mathbb{R}^3$  (these vector fields define the kinematic model of a simplified car). At  $p = 0$ , the coordinates  $(x, \theta)$  have order 1 and  $y$  has order 2. Since the weights at  $p$  are  $(1, 1, 2)$  (just compute the bracket  $[X_1, X_2]$ ),  $(x, \theta, y)$  is a system of privileged coordinates at 0. Taking the Taylor expansion of  $X_1$  and  $X_2$  in the latter coordinates, we obtain as homogeneous components:

$$X_1^{(-1)} = \partial_x + \theta \partial_y, \quad X_1^{(0)} = 0, \quad X_1^{(1)} = -\frac{\theta^2}{2} \partial_x - \frac{\theta^3}{3!} \partial_y, \quad \dots$$

and  $X_2^{(-1)} = X_2 = \partial_\theta$ . Thus the homogeneous nilpotent approximation of  $(X_1, X_2)$  at 0 in coordinates  $(x, \theta, y)$  is:

$$\widehat{X}_1 = \partial_x + \theta \partial_y, \quad \widehat{X}_2 = \partial_\theta.$$

We check easily that the Lie brackets of length 3 of these vectors are zero:  $[\widehat{X}_1, [\widehat{X}_1, \widehat{X}_2]] = [\widehat{X}_2, [\widehat{X}_1, \widehat{X}_2]] = 0$ , so the Lie algebra  $\text{Lie}\{\widehat{X}_1, \widehat{X}_2\}$  is nilpotent of step 2.

The homogeneous nilpotent approximation is not intrinsic to the frame  $(X_1, \dots, X_m)$ , it depends on the chosen system of privileged coordinates. However, if  $\widehat{X}_1, \dots, \widehat{X}_m$  and  $\widehat{X}'_1, \dots, \widehat{X}'_m$  are the nilpotent approximations associated to two different systems of coordinates, then their Lie algebras  $\text{Lie}(\widehat{X}_1, \dots, \widehat{X}_m)$  and  $\text{Lie}(\widehat{X}'_1, \dots, \widehat{X}'_m)$  are isomorphic. If moreover  $p$  is a regular point, then  $\text{Lie}(\widehat{X}_1, \dots, \widehat{X}_m)$  is isomorphic to the graded nilpotent Lie algebra

$$\text{Gr}(\Delta)_p = \Delta(p) \oplus (\Delta^2/\Delta^1)(p) \oplus \dots \oplus (\Delta^{r-1}/\Delta^r)(p).$$

*Remark.* The nilpotent approximation denotes in fact two different objects. Each  $\widehat{X}_i$  can be seen as a vector field on  $\mathbb{R}^n$  or as the representation in  $z$  coordinates of the vector field  $z^* \widehat{X}_i$  defined on a neighbourhood of  $p$  in  $M$ . It will cause no confusion since the nilpotent approximation is associated to a given system of privileged coordinates.

It is worth to notice the particular form of the nilpotent approximation in privileged coordinates. Indeed, write  $\widehat{X}_i = \sum_{j=1}^n f_{ij}(z) \partial_{z_j}$ . Since  $\widehat{X}_i$  is homogeneous of degree  $-1$  and  $\partial_{z_j}$  of degree  $-w_j$ , the function  $f_{ij}$  is a homogeneous polynomial of weighted degree  $w_j - 1$ . In particular it can not involve variables of weight greater than  $w_j - 1$ , that is,

$$\widehat{X}_i(z) = \sum_{j=1}^n f_{ij}(z_1, \dots, z_{n_{w_j-1}}) \partial_{z_j}.$$

The nonholonomic control system  $\dot{z} = \sum_{i=1}^m u_i \widehat{X}_i(z)$  associated to the nilpotent approximation is then polynomial and in a triangular form:

$$\dot{z}_j = \sum_{i=1}^m u_i f_{ij}(z_1, \dots, z_{n_{w_j-1}}).$$

Such a form is “easy” to integrate: given the input function  $(u_1(t), \dots, u_m(t))$ , it is possible to compute the coordinates  $z_j$  one after the other, only by computing primitives.

As vector fields on  $\mathbb{R}^n$ ,  $\widehat{X}_1, \dots, \widehat{X}_m$  generate a sub-Riemannian distance on  $\mathbb{R}^n$  which is homogeneous with respect to the dilation  $\delta_t$ .

**Lemma 3.5.**

- (i) The family  $(\widehat{X}_1, \dots, \widehat{X}_m)$  satisfies Chow's Condition on  $\mathbb{R}^n$ .
- (ii) The growth vector of  $(\widehat{X}_1, \dots, \widehat{X}_m)$  at 0 is equal to the one of  $(X_1, \dots, X_m)$  at  $p$ .

Denote by  $\widehat{d}$  the sub-Riemannian distance on  $\mathbb{R}^n$  associated to  $(\widehat{X}_1, \dots, \widehat{X}_m)$ .

- (iii) The distance  $\widehat{d}$  is homogeneous of degree 1:

$$\widehat{d}(\delta_t x, \delta_t y) = t \widehat{d}(x, y).$$

- (iv) There exists a constant  $C > 0$  such that, for all  $z \in \mathbb{R}^n$ ,

$$\frac{1}{C} \|z\| \leq \widehat{d}(0, z) \leq C \|z\|.$$

*Proof.* Through the coordinates  $z$  we identify the neighbourhood  $U$  of  $p$  in  $M$  with a neighbourhood of 0 in  $\mathbb{R}^n$ .

For an iterated bracket  $X_I = [X_{i_k}, \dots, [X_{i_2}, X_{i_1}]]$  of the vector fields  $X_1, \dots, X_m$ , we denote by  $\widehat{X}_I = [\widehat{X}_{i_k}, \dots, [\widehat{X}_{i_2}, \widehat{X}_{i_1}]]$  the corresponding bracket of the vector fields  $\widehat{X}_1, \dots, \widehat{X}_m$ , and for  $k \geq 1$  we set  $\widehat{\Delta}^k = \text{span}\{\widehat{X}_I : |I| \leq k\}$ . As noticed in the proof of Proposition 3.4, a bracket  $\widehat{X}_I$  of length  $|I| = k$  is homogeneous of weighted degree  $-k$ , and by construction of the nilpotent approximation, there holds  $X_I = \widehat{X}_I +$  terms of order  $> -k$ . It implies

$$\widehat{X}_I(0) = X_I(p) \bmod \text{span}\{\partial_{z_j} \big|_p : w_j < k\} = X_I(p) \bmod \Delta^{k-1}(p).$$

As a consequence, for any integer  $k \geq 1$ , we have

$$\dim \widehat{\Delta}^k(0) = \dim \Delta^k(p). \tag{10}$$

Moreover, if  $X_{I_1}, \dots, X_{I_n}$  form an adapted frame at  $p$ , then  $\text{rank}(\widehat{X}_{I_1}, \dots, \widehat{X}_{I_n})(0) = n$ , which implies that the determinant of  $(\widehat{X}_{I_1}(0), \dots, \widehat{X}_{I_n}(0))$  is nonzero. Since this determinant is an homogeneous polynomial of weighted degree 0, it is nonzero everywhere, and then  $\text{rank}(\widehat{Y}_1, \dots, \widehat{Y}_n) = n$  on the whole  $\mathbb{R}^n$ , which gives (i). Property (ii) then results from (10).

As for the property (iii), consider the nonholonomic system  $\dot{z} = \sum_{i=1}^m u_i \widehat{X}_i(z)$ . Observe that, if  $\widehat{\gamma}$  is a trajectory of this system, that is if

$$\dot{\widehat{\gamma}}(t) = \sum_{i=1}^m u_i \widehat{X}_i(\widehat{\gamma}(t)), \quad t \in [0, T],$$

then the dilated curve  $\delta_\lambda \widehat{\gamma}$  satisfies

$$\frac{d}{dt} \delta_\lambda \widehat{\gamma}(t) = \sum_{i=1}^m \lambda u_i \widehat{X}_i(\delta_\lambda \widehat{\gamma}(t)), \quad t \in [0, T].$$

Thus  $\delta_\lambda \widehat{\gamma}$  is a trajectory of the same system, with extremities  $(\delta_\lambda \widehat{\gamma})(0) = \delta_\lambda(\widehat{\gamma}(0))$  and  $(\delta_\lambda \widehat{\gamma})(T) = \delta_\lambda(\widehat{\gamma}(T))$ , and its length equals  $\lambda \text{length}(\widehat{\gamma})$ . This proves the homogeneity of  $\widehat{d}$ .

Finally, since  $(\widehat{X}_1, \dots, \widehat{X}_m)$  satisfies Chow's Condition, the distance  $\widehat{d}(0, \cdot)$  is continuous on  $\mathbb{R}^n$  (see Corollary 2.4). We can then choose a real number  $C > 0$  such that, on the compact set  $\{\|z\| = 1\}$ , we have  $1/C \leq \widehat{d}(0, z) \leq C$ . Both functions  $\widehat{d}(0, z)$  and  $\|z\|$  being homogeneous of degree 1, the inequality of Property (iv) follows.  $\square$

### 3.4 Distance estimates

As it is the case for Riemannian distances, it is impossible – except in very specific cases – to compute analytically a sub-Riemannian distance (it would require to determine all minimizing curves). It is then very important to obtain estimates of the distance, at least locally. In a Riemannian manifold  $(M, g)$ , the situation is rather simple: in local coordinates  $x$  centered at a point  $p$ , the Riemannian distance  $d_R$  satisfies:

$$d_R(q, q') = \|x(q) - x(q')\|_{g_p} + o(\|x(q)\|_{g_p} + \|x(q')\|_{g_p}),$$

where  $\|\cdot\|_{g(p)}$  is the Euclidean norm induced by the value  $g_p$  of the metric  $g$  at  $p$ . This formula has two consequences: first, it shows that the Riemannian distance behaves at the first-order as the Euclidean distance associated to  $\|\cdot\|_{g(p)}$ ; secondly, the norm  $\|\cdot\|_{g(p)}$  gives explicit estimates of  $d_R$  near  $p$ , such as:

$$\frac{1}{C} \|x(q)\|_{g_p} \leq d_R(p, q) \leq C \|x(q)\|_{g_p}.$$

In sub-Riemannian geometry, two such properties hold, but do not depend on the same function: the first-order behaviour near  $p$  is characterized by the distance  $\widehat{d}_p$  of a nilpotent approximation at  $p$ , whence explicit local estimates of  $d(p, \cdot)$  are given by the pseudo-norm  $\|\cdot\|_p$ , as stated below.

**Theorem 3.6.** *The following statement holds if and only if  $z_1, \dots, z_n$  are privileged coordinates at  $p$ :*

*there exist constants  $C_p$  and  $\varepsilon_p > 0$  such that, if  $d(p, q^z) < \varepsilon_p$ , then*

$$\frac{1}{C_p} \|z\| \leq d(p, q^z) \leq C_p \|z\| \tag{11}$$

(as usual,  $q^z$  denotes the point near  $p$  with coordinates  $z$ ).

**Corollary 3.7** (Ball-Box Theorem). *Expressed in a given system of privileged coordinates, the sub-Riemannian balls  $B(p, \varepsilon)$  satisfy, for  $\varepsilon < \varepsilon_p$ ,*

$$\text{Box}\left(\frac{1}{C_p}\varepsilon\right) \subset B(p, \varepsilon) \subset \text{Box}(C_p\varepsilon),$$

where  $\text{Box}(\varepsilon) = [-\varepsilon^{w_1}, \varepsilon^{w_1}] \times \cdots \times [-\varepsilon^{w_n}, \varepsilon^{w_n}]$ .

*Remark.* The constants  $C_p$  and  $\varepsilon_p$  depend on the base point  $p$ . Around a regular point  $p_0$ , it is possible to construct systems of privileged coordinates depending continuously on the base point  $p$ . In this case, the corresponding constants  $C_p$  and  $\varepsilon_p$  depend continuously on  $p$ . This is not true at a singular point. In particular, if  $p_0$  is singular, the estimate (11) does not hold uniformly near  $p_0$ : we can not choose the constants  $C_p$  and  $\varepsilon_p$  independently on  $p$  near  $p_0$  (see section 4.2 for uniform versions of Ball-Box Theorem).

Ball-Box Theorem is stated in different papers, often under the hypothesis that the point  $p$  is regular. As far as I know, two valid proofs exist: in [NSW85] and in [Bel96]. The result also appears without proof in [Gro96] and in [Ger84], and with erroneous proofs in [Mit85] and in [Mon02].

We present here a proof adapted from the one of Bellaïche (our is much simpler because Bellaïche actually proves a more general result, namely (14)). Basically, the idea is to compare the distances  $d$  and  $\hat{d}$ . The main step is Lemma 3.8 below, which is essential to explain the role of nilpotent approximations in control theory.

Observe first that, by the definition of order, a system of coordinates  $z$  is privileged if and only if  $d(p, q^z) \geq Cst \|z\|$ . What remains to prove is that, if  $z$  are privileged coordinates, then  $d(p, q^z) \leq Cst \|z\|$ .

Fix a point  $p \in M$ , and a system of privileged coordinates at  $p$ . Through these coordinates we identify a neighbourhood of  $p$  in  $M$  with a neighbourhood of 0 in  $\mathbb{R}^n$ . As in the preceding subsection, we denote by  $\hat{X}_1, \dots, \hat{X}_m$  the homogeneous nilpotent approximation of  $X_1, \dots, X_m$  at  $p$  (associated to the given system of privileged coordinates) and by  $\hat{d}$  the induced sub-Riemannian distance on  $\mathbb{R}^n$ . Recall also that  $r = w_n$  denotes the degree of nonholonomy at  $p$ .

**Lemma 3.8.** *There exist constants  $C$  and  $\varepsilon > 0$  such that, for any  $x_0 \in \mathbb{R}^n$  and any  $t \in \mathbb{R}^+$  with  $\tau = \max(\|x_0\|, t) < \varepsilon$ , we have*

$$\|x(t) - \hat{x}(t)\| \leq C\tau t^{1/r},$$



where  $x(\cdot)$  and  $\hat{x}(\cdot)$  are trajectories of the nonholonomic systems associated respectively to  $X_1, \dots, X_m$  and  $\hat{X}_1, \dots, \hat{X}_m$ , starting at the same point  $x_0$ , defined by the same control function  $u(\cdot)$ , and with velocity one (i.e.  $\sum_i u_i^2 \equiv 1$ ).

*Proof.* The first step is to prove that there exists a constant such that  $\|x(t)\|$  and  $\|\hat{x}(t)\| \leq Cst\tau$  for small enough  $\tau$ . Let us do it for  $x(t)$ , the proof is exactly the same for  $\hat{x}(t)$ .

The equation of a trajectory of the control system associated to  $X_1, \dots, X_m$  is

$$\dot{x}_j = \sum_{i=1}^m u_i (f_{ij}(x) + r_{ij}(x)), \quad j = 1, \dots, n,$$

where  $f_{ij}(x) + r_{ij}(x)$  is of order  $\leq w_j - 1$  at 0. Thus there exist a constant such that, when  $\|x\|$  is small enough,  $|f_{ij}(x) + r_{ij}(x)| \leq Cst \|x\|^{w_j-1}$  for any  $j = 1, \dots, n$  and any  $i = 1, \dots, m$ . Note that, along a trajectory starting at  $x_0$ ,  $\|x\|$  is small when  $\tau$  is. If moreover the trajectory has velocity one, which is the case here, we obtain:

$$|\dot{x}_j| \leq Cst \|x\|^{w_j-1}. \quad (12)$$

To integrate this inequality, choose an integer  $N$  such that all  $N/w_j$  are even integers and set  $|||x||| = \left(\sum_{i=1}^n x_i^{N/w_i}\right)^{1/N}$ . The function  $|||x|||$  is equivalent to  $\|x\|$  in the norm sense and is differentiable out of the origin. Inequality (12) implies  $\frac{d}{dt}|||x||| \leq Cst$ , and then, by integration,

$$|||x(t)||| \leq Cst \times t + |||x(0)||| \leq Cst \times \tau.$$

The pseudo-norms  $|||x|||$  and  $\|x\|$  being equivalent, we obtain, for a trajectory starting at  $x_0$ ,  $\|x(t)\| \leq Cst \times \tau$  when  $\tau$  is small enough.

The second step is to prove  $|x_j(t) - \hat{x}_j(t)| \leq Cst \tau^{w_j} t$ . The function  $x_j - \hat{x}_j$  satisfies the differential equation

$$\begin{aligned} \dot{x}_j - \dot{\hat{x}}_j &= \sum_{i=1}^m u_i (f_{ij}(x) - f_{ij}(\hat{x}) + r_{ij}(x)), \\ &= \sum_{i=1}^m u_i \left( \sum_{\{k: w_k < w_j\}} (x_k - \hat{x}_k) Q_{ijk}(x, \hat{x}) + r_{ij}(x) \right), \end{aligned}$$

where  $Q_{ijk}(x, \hat{x})$  is a homogeneous polynomial of weighted degree  $w_j - w_k - 1$ . For  $\|x\|$  and  $\|\hat{x}\|$  small enough, we have

$$|r_{ij}(x)| \leq Cst \|x\|^{w_j} \quad \text{and} \quad |Q_{ijk}(x, \hat{x})| \leq Cst (\|x\| + \|\hat{x}\|)^{w_j - w_k - 1}.$$

By using the inequalities of the first step, we obtain finally, for  $\tau$  small enough,

$$|\dot{x}_j(t) - \dot{\hat{x}}_j(t)| \leq Cst \sum_{\{k: w_k < w_j\}} |x_k(t) - \hat{x}_k(t)| \tau^{w_j - w_k - 1} + Cst \tau^{w_j}. \quad (13)$$

This system of inequalities has a triangular form, it can be integrated iteratively. For  $w_j = 1$ , the inequality is  $|\dot{x}_j(t) - \dot{\hat{x}}_j(t)| \leq Cst \tau$ , and so  $|x_j(t) - \hat{x}_j(t)| \leq Cst \tau t$ . By induction, let  $j > n_1$  and assume  $|x_k(t) - \hat{x}_k(t)| \leq Cst \tau^{w_k} t$  for  $k < j$ . Inequality (13) implies

$$|\dot{x}_j(t) - \dot{\hat{x}}_j(t)| \leq Cst \tau^{w_j - 1} t + Cst \tau^{w_j} \leq Cst \tau^{w_j},$$

and so  $|x_j(t) - \hat{x}_j(t)| \leq Cst \tau^{w_j} t$ .

Finally,

$$\|x(t) - \hat{x}(t)\| \leq Cst \tau (t^{1/w_1} + \dots + t^{1/w_n}) \leq Cst \tau t^{1/r},$$

which completes the proof of the lemma.  $\square$

*Proof of Theorem 3.6.* We will show that, for  $\|x^0\|$  small enough,

$$d(0, x^0) \leq 2\hat{d}(0, x^0),$$

and so  $d(0, x^0) \leq Cst \|x^0\|$  by Lemma 3.5. As noticed earlier, this proves Theorem 3.6.

Fix  $x^0 \in \mathbb{R}^n$ ,  $\|x^0\| < \varepsilon$ . Let  $\hat{x}_0(t)$ ,  $t \in [0, T_0]$ , be a minimizing curve for  $\hat{d}$ , having velocity one, and joining  $x^0$  to 0. Let  $x_0(t)$ ,  $t \in [0, T_0]$ , be the trajectory of the control system associated to  $X_1, \dots, X_m$  starting at  $x^0$  and defined by the same control function than  $\hat{x}_0(t)$ . Set  $x^1 = x_0(T_0)$ .

Thus  $T_0 = \hat{d}(0, x^0) = \text{length}(x_0(\cdot))$ . Moreover, by Lemma 3.8,

$$\|x^1\| = \|x_0(T_0) - \hat{x}_0(T_0)\| \leq C\tau T_0^{1/r},$$

where  $\tau = \max(\|x^0\|, T_0)$ . By Lemma 3.5,  $T_0 = \hat{d}(0, x^0)$  satisfies  $T_0 \geq \|x^0\|/C'$ , so  $\tau \leq C'T_0$ , and

$$\hat{d}(0, x^1) \leq C'\|x^1\| \leq C''\hat{d}(0, x^0)^{1+1/r},$$

with  $C'' = C'^2C$ .

Choose now  $\hat{x}_1(t)$ ,  $t \in [0, T_1]$ , a minimizing curve for  $\hat{d}$ , having velocity one, and joining  $x^1$  to 0. Let  $x_1(t)$ ,  $t \in [0, T_1]$ , be the trajectory of the control system associated to  $X_1, \dots, X_m$  starting at  $x^1$  and defined by the

same control function than  $\widehat{x}_1(t)$ . Set  $x^2 = x_1(T_1)$ . As previously, we have  $\text{length}(x_1(\cdot)) = \widehat{d}(0, x^1)$  and  $\widehat{d}(0, x^2) \leq C''\widehat{d}(0, x^1)^{1+1/r}$ .

Iterating this construction, we obtain a sequence  $x^0, x^1, x^2, \dots$  of points such that  $\widehat{d}(0, x^{k+1}) \leq C''\widehat{d}(0, x^k)^{1+1/r}$ , and trajectories  $x_k(\cdot)$  joining  $x^k$  to  $x^{k+1}$  of length equal to  $\widehat{d}(0, x^k)$ .

By taking  $\|x^0\|$  small enough, we can assume  $C''\widehat{d}(0, x^0)^{1/r} \leq 1/2$ . We have then  $\widehat{d}(0, x^1) \leq \widehat{d}(0, x^0)/2$ ,  $\dots$ ,  $\widehat{d}(0, x^k) \leq \widehat{d}(0, x^0)/2^k$ ,  $\dots$  As a consequence,  $x^k$  tends to 0 as  $k \rightarrow +\infty$ , and putting end to end the curves  $x_k(\cdot)$ , we obtain a trajectory joining  $x^0$  to 0 of length  $\widehat{d}(0, x^0) + \widehat{d}(0, x^1) + \dots \leq 2\widehat{d}(0, x^0)$ . This implies  $d(0, x^0) \leq 2\widehat{d}(0, x^0)$ , and so the theorem.  $\square$

### 3.5 Approximate motion planning

Given a control system  $(\Sigma)$ , the *motion planning problem* is to steer  $(\Sigma)$  from an initial point to a final point. For nonholonomic systems, the exact problem is in general unsolvable. However methods exist for particular class of system, in particular for nilpotent (or nilpotentizable) systems. It is then of interest to devise approximate motion planning techniques based on nilpotent approximations. These techniques are Newton type methods, the nilpotent approximation playing the role of the usual linearization.

Precisely, consider a nonholonomic control system

$$(\Sigma) : \quad \dot{x} = \sum_{i=1}^m u_i X_i(x), \quad x \in \mathbb{R}^n,$$

and initial and final points  $a$  and  $b$  in  $\mathbb{R}^n$ . Denote by  $\widehat{X}_1, \dots, \widehat{X}_m$  a nilpotent approximation of  $X_1, \dots, X_m$  at  $b$ . The  $k$ -step of an approximate motion planning algorithm take the following form ( $x^k$  denotes the state of the system,  $x^0$  being the initial point  $a$ ):

1. compute a control law  $u(t)$ ,  $t \in [0, T]$ , steering the control system associated to  $\widehat{X}_1, \dots, \widehat{X}_m$  from  $x^k$  to  $b$ ;
2. compute the trajectory  $x(\cdot)$  of  $(\Sigma)$  with control law  $u(\cdot)$  starting from  $x^k$ ;
3. set  $x^{k+1} = x(T)$ .

The question is the following: is this algorithm convergent? or, at least, locally convergent? The answer to the latter question will be positive, but we need an extra hypothesis on the control law given in point 2 of the algorithm, namely,

(H) there exists a constant  $K$  such that, if  $x^k$  and  $b$  are close enough, then

$$\int_0^T \sqrt{\sum u_i^2(t)} dt \leq K \widehat{d}(b, x_k).$$

Note that a control corresponding to a minimizing curve for  $\widehat{d}$  satisfies this condition. Other standard methods using Lie groups (like the one in [LS91]) or based on the triangular form of the homogeneous nilpotent approximation satisfy also this hypothesis.

We can also assume without restriction that the control is normalized, that is,  $\sum_{i=1}^m u_i^2(t) \equiv 1$ .

The local convergence is then proved in exactly the same way than Theorem 3.6: from Lemmas 3.8 and 3.5, we have  $\widehat{d}(b, x^{k+1}) \leq C'' T^{1+1/r}$ , and using hypothesis (H), we obtain

$$\widehat{d}(b, x^{k+1}) \leq C'' K^{1+1/r} \widehat{d}(b, x_k)^{1+1/r}.$$

If  $a$  is close enough to  $b$ , we have, at each step of the algorithm,  $\widehat{d}(b, x^{k+1}) \leq \widehat{d}(b, x_k)/2$ , which proves the local convergence of the algorithm, that is: *for each point  $b \in M$ , there exists a constant  $\varepsilon_b > 0$  such that, if  $d(a, b) < \varepsilon_b$ , then the approximate motion planning algorithm steering the system from  $a$  to  $b$  converges.*

To obtain a globally convergent algorithm, a natural idea is to iterate the locally convergent one. This requires the construction of a finite sequence of intermediate goals  $b_0 = a, b_1, \dots, b_N = b$  such that  $d(b_{i-1}, b_i) < \varepsilon_{b_i}$ . However the constant  $\varepsilon_b$  depends on  $b$  and, as already noticed for Theorem 3.6, it is not possible to have a uniform nonzero constant near singular points. Thus this method gives a globally convergent algorithm only when every point is regular.

## 4 Tangent structure

Consider a manifold  $M$  endowed with a sub-Riemannian distance  $d$  on  $M$ . The so-defined metric space  $(M, d)$  is called a *Carnot-Carathéodory space*. The object of this section is to describe the local structure of this metric space.

### 4.1 Metric tangent space

A notion of tangent space can be defined for a general metric space. Indeed, in describing the tangent space to a manifold, we usually imagine looking

at smaller and smaller neighbourhoods of a given point, the manifold being fixed. Equivalently, we can imagine looking at a fixed neighbourhood, but expanding the manifold. As noticed by Gromov, this idea can be used for metric spaces.

If  $\mathsf{X}$  is a metric space with distance  $d$ , we define  $\lambda\mathsf{X}$ , for  $\lambda > 0$ , as the metric space with same underlying set than  $\mathsf{X}$  and distance  $\lambda d$ . A *pointed metric space*  $(\mathsf{X}, x)$  is a metric space with a distinguished point  $x$ .

Loosely speaking, a metric tangent space to the metric space  $\mathsf{X}$  at  $x$  is a pointed metric space  $(C_x\mathsf{X}, y)$  such that

$$(C_x\mathsf{X}, y) = \lim_{\lambda \rightarrow +\infty} (\lambda\mathsf{X}, x).$$

Of course, for this definition to make sense, we have to define the limit of pointed metric spaces.

Let us first define the Gromov-Hausdorff distance between metric spaces. Recall that, in a metric space  $\mathsf{X}$ , the Hausdorff distance  $\text{H-dist}(A, B)$  between two subsets  $A$  and  $B$  of  $\mathsf{X}$  is the infimum of  $\rho$  such that any point of  $A$  is within a distance  $\rho$  of  $B$  and any point of  $B$  is within a distance  $\rho$  of  $A$ . The Gromov-Hausdorff distance  $\text{GH-dist}(\mathsf{X}, \mathsf{Y})$  between two metric spaces  $\mathsf{X}$  and  $\mathsf{Y}$  is the infimum of Hausdorff distances  $\text{H-dist}(i(\mathsf{X}), j(\mathsf{Y}))$  over all metric spaces  $\mathsf{Z}$  and all isometric embeddings  $i : \mathsf{X} \rightarrow \mathsf{Z}$ ,  $j : \mathsf{Y} \rightarrow \mathsf{Z}$ .

Thanks to Gromov-Hausdorff distance, one can define the notion of limit of a sequence of pointed metric spaces:  $(\mathsf{X}_n, x_n)$  converge to  $(\mathsf{X}, x)$  if, for any positive  $r$ ,

$$\text{GH-dist}(B^{\mathsf{X}_n}(x_n, r), B^{\mathsf{X}}(x, r)) \rightarrow 0 \quad \text{as } n \rightarrow +\infty$$

where  $B^{\mathsf{Y}}(y, r)$  is considered as a metric space, endowed with the distance of  $\mathsf{Y}$ . Note that all pointed metric spaces isometric to  $(\mathsf{X}, x)$  are also limit of  $(\mathsf{X}_n, x_n)$ . However the limit is unique up to an isometry provided the closed balls around the distinguished point are compact.

Finally, one says that  $(\mathsf{X}_\lambda, x_\lambda)$  converge to  $(\mathsf{X}, x)$  when  $\lambda \rightarrow \infty$  if, for every sequence  $\lambda_n$ ,  $(\mathsf{X}_{\lambda_n}, x_{\lambda_n})$  converge to  $(\mathsf{X}, x)$ .

**Definition.** A pointed metric space  $(C_x\mathsf{X}, y)$  is a *metric tangent space* to the metric space  $\mathsf{X}$  at  $x$  if  $(\lambda\mathsf{X}, x)$  converge to  $(C_x\mathsf{X}, y)$  as  $\lambda \rightarrow +\infty$ . If it exists, it is unique up to an isometry.

For a Riemannian metric space  $(M, d_R)$  induced by a Riemannian metric  $g$  on a manifold  $M$ , metric tangent spaces at a point  $p$  exist and are isometric to the Euclidean space  $(T_pM, g_p)$ , that is, the standard tangent space endowed with the scalar product defined by the quadratic form  $g_p$ .

For the Carnot-Carathéodory space  $(M, d)$  defined by a sub-Riemannian manifold, the metric tangent space is given by the nilpotent approximation.

**Theorem 4.1.** *A Carnot-Carathéodory space  $(M, d)$  admits metric tangent spaces  $(C_p M, y)$  at every point  $p \in M$ . The space  $C_p M$  is a Carnot-Carathéodory space isometric to  $(\mathbb{R}^n, \widehat{d})$ , where  $\widehat{d}$  is the sub-Riemannian distance associated to a homogeneous nilpotent approximation at  $p$ .*

This theorem is a consequence of a strong version of Theorem 3.6 established by Bellaïche: there exists  $C > 0$  such that, for  $q$  and  $q'$  in a neighbourhood of  $p$ ,

$$|d(q, q') - \widehat{d}(q, q')| \leq C \widehat{d}(p, q) d(q, q')^{1/r}. \quad (14)$$

Intrinsic characterization (*i.e.* up to a unique isometry) of the metric tangent space can be found in [MM00] and [FJ03].

What is the algebraic structure of  $C_p M$ ? Of course it is not a linear space in general: for instance,  $\widehat{d}$  is homogeneous of degree 1 but with respect to dilations  $\delta_t$  and not to the usual Euclidean dilations. We will see that  $C_p M$  has a natural structure of group, or at least of quotient of groups.

Denote by  $G_p$  the group generated by the diffeomorphisms  $\exp(t\widehat{X}_i)$  acting on the left on  $\mathbb{R}^n$ . Since  $\mathfrak{g}_p = \text{Lie}(\widehat{X}_1, \dots, \widehat{X}_m)$  is a nilpotent Lie algebra,  $G_p = \exp(\mathfrak{g}_p)$  is a simply connected group, having  $\mathfrak{g}_p$  as its Lie algebra.

This Lie algebra  $\mathfrak{g}_p$  splits into homogeneous components

$$\mathfrak{g}_p = \mathfrak{g}^{-1} \oplus \dots \oplus \mathfrak{g}^{-r},$$

where  $\mathfrak{g}^{-s}$  is the set of homogeneous vector fields of degree  $-s$ , and so  $\mathfrak{g}_p$  is a graded Lie algebra. The first component  $\mathfrak{g}^{-1} = \text{span}\langle \widehat{X}_1, \dots, \widehat{X}_m \rangle$  generates  $\mathfrak{g}_p$  as a Lie algebra. All these properties imply that  $G_p$  is a Carnot group.

**Definition.** A *Carnot group* is a simply connected Lie group whose Lie algebra is graded, nilpotent, and generated by its first component.

Let  $\widehat{\xi}_1, \dots, \widehat{\xi}_m$  be the elements  $\widehat{X}_1, \dots, \widehat{X}_m$  of  $\mathfrak{g}_p$  viewed as right-invariant vector fields on  $G_p$ , that is

$$\widehat{\xi}(g) = \frac{d}{dt} [\exp(t\widehat{X}_i)g] \Big|_{t=0}.$$

To  $(\widehat{\xi}_1, \dots, \widehat{\xi}_m)$  is associated a right-invariant sub-Riemannian metric and a sub-Riemannian distance  $d_{G_p}$  on  $G_p$ .

The action of  $G_p$  on  $\mathbb{R}^n$  is transitive, since  $(\widehat{X}_1, \dots, \widehat{X}_m)$  satisfies Chow's Condition on  $\mathbb{R}^n$ : the orbit of 0 under the action of  $G_p$  is the whole  $\mathbb{R}^n$ . The mapping  $\phi_p : G_p \rightarrow \mathbb{R}^n$ ,  $\phi_p(g) = g(0)$ , is then surjective.

Case  $p$  regular.

**Proposition 4.2.** *If  $p$  is a regular point, then  $\dim G_p = n$ .*

*Proof.* Let  $X_{I_1}, \dots, X_{I_n}$  be an adapted frame at  $p$ . Due to the regularity of  $p$ , it is also an adapted frame near  $p$ , so any bracket  $X_J$  can be written as

$$X_J(z) = \sum_{\{i: |I_i| \leq |J|\}} a_i(z) X_{I_i}(z),$$

where each  $a_i$  is a function of order  $\geq |I_i| - |J|$ . By taking the homogeneous terms of degree  $-|J|$  in this expression, we obtain

$$\widehat{X}_J(z) = \sum_{\{i: |I_i|=|J|\}} a_i(0) \widehat{X}_{I_i}(z),$$

and so  $\widehat{X}_J \in \text{span}\langle \widehat{X}_{I_1}, \dots, \widehat{X}_{I_n} \rangle$ . Thus  $\widehat{X}_{I_1}, \dots, \widehat{X}_{I_n}$  is a basis of  $\mathfrak{g}_p$ , and so  $\dim G_p = n$ .  $\square$

If  $p$  is regular, the mapping  $\phi_p$  is a diffeomorphism. Moreover  $\phi_{p*} \widehat{\xi}_i = \widehat{X}_i$ , that is  $\phi_p$  maps the sub-Riemannian metric associated to  $(\widehat{\xi}_1, \dots, \widehat{\xi}_m)$  to the one associated to  $(\widehat{X}_1, \dots, \widehat{X}_m)$ .

**Lemma 4.3.** *When  $p$  is a regular point, the metric tangent space  $C_p M$  and the Carnot-Carathéodory space  $(\mathbb{R}^n, \widehat{d})$  are isometric to the Carnot group  $G_p$  endowed with the right-invariant sub-Riemannian distance  $d_{G_p}$ .*

Carnot groups are to sub-Riemannian geometry as Euclidean spaces are to Riemannian geometry: the internal operation – addition – is replaced by the law group and the external operation – product by a real number – by the dilations. Indeed, recall that the dilations  $\delta_t$  act on  $\mathfrak{g}_p$  as a multiplication by  $t^{-s}$  on  $\mathfrak{g}^{-s}$ ; it extends to  $G_p$  by the exponential mapping. Notice that, when  $G_p$  is Abelian (*i.e.* commutative) it has a linear structure and the sub-Riemannian metric on  $G_p$  is a Euclidean metric.

General case.

Without hypothesis on  $p$ ,  $G_p$  may be of dimension greater than  $n$  and when it is the case the map  $\phi_p$  is not injective. Denoting by  $H_p = \{g \in G_p : g(0) = 0\}$  the isotropy subgroup of 0,  $\phi_p$  induces a diffeomorphism

$$\psi_p : G_p/H_p \rightarrow \mathbb{R}^n.$$

Beware:  $G_p/H_p$  is in general only a coset space, not a quotient group.

Observe that  $H_p$  is invariant under dilations, since  $\delta_t g(\delta_t x) = \delta_t(g(x))$ . Hence  $H_p$  is connected and simply connected, and so  $H_p = \exp(\mathfrak{h}_p)$ , where  $\mathfrak{h}_p$  is the Lie sub-algebra of  $\mathfrak{g}_p$  containing the vector fields vanishing at 0:

$$\mathfrak{h}_p = \{Z \in \mathfrak{g}_p : Z(0) = 0\}.$$

As  $\mathfrak{g}_p$ ,  $\mathfrak{h}_p$  is invariant under dilations and splits into homogeneous components.

Now, the elements  $\widehat{X}_1, \dots, \widehat{X}_m$  of  $\mathfrak{g}_p$  acts on the left on  $G_p/H_p = \{gH_p : g \in G_p\}$  under the denomination  $\bar{\xi}_1, \dots, \bar{\xi}_m$ :

$$\bar{\xi}_i(g) = \frac{d}{dt} [\exp(t\widehat{X}_i)gH_p] |_{t=0}.$$

These vector fields define a sub-Riemannian metric and a sub-Riemannian distance  $\bar{d}$  on  $G_p/H_p$ . We clearly have  $\psi_{p*}\bar{\xi}_i = \widehat{X}_i$ , so  $\psi_p$  maps the sub-Riemannian metric associated to  $(\bar{\xi}_1, \dots, \bar{\xi}_m)$  to the one associated to  $(\widehat{X}_1, \dots, \widehat{X}_m)$ .

**Theorem 4.4.** *The metric tangent space  $C_pM$  and  $(\mathbb{R}^n, \widehat{d})$  are isometric to the homogeneous space  $G_p/H_p$  endowed with the sub-Riemannian distance  $\bar{d}$ .*

## 4.2 Desingularization and uniform distance estimate

To get rid of singular points, the usual way is to consider a singularity as the projection of a regular object. The algebraic structure of the metric tangent space yields a good way of lifting and projecting Carnot-Carathéodory spaces. We start with nilpotent approximations.

We keep the notations and definitions of the preceding section. At a singular point  $p$ , we have the following diagram:

$$\begin{array}{ccc} (G_p, d_{G_p}) & & \\ \pi \downarrow & \phi_p \searrow & \\ (G_p/H_p, \bar{d}) & \xrightarrow{\psi_p} & (\mathbb{R}^n, \widehat{d}) \end{array}$$

Since the sub-Riemannian metric on  $G_p$  is a right-invariant, every point in the space  $(G_p, d_{G_p})$  is regular. We say that the space  $(G_p, d_{G_p})$  is a *equiregular*. Thus  $(\mathbb{R}^n, \widehat{d})$  is the projection of an equiregular space.

Recall that  $\widehat{\xi}_1, \dots, \widehat{\xi}_m$  (resp.  $\bar{\xi}_1, \dots, \bar{\xi}_m$ ) is mapped to  $\widehat{X}_1, \dots, \widehat{X}_m$  by  $\phi_p$  (resp.  $\psi_p$ ). Working in a system of coordinates, we identify  $G_p/H_p$  with  $\mathbb{R}^n$  and  $\bar{\xi}_i$  with  $\widehat{X}_i$ . These coordinates  $x$  on  $\mathbb{R}^n \simeq G_p/H_p$  induce coordinates  $(x, z) \in \mathbb{R}^N$  on  $G_p$  and we have:

$$\widehat{\xi}_i(x, z) = \widehat{X}_i(x) + \sum_{j=n+1}^N b_{ij}(x, z) \partial_{z_j}. \quad (15)$$

Consider a trajectory  $(x(t), z(t))$  in  $G_p$  associated to the control  $u(t)$ , that is

$$(\dot{x}(t), \dot{z}(t)) = \sum_{i=1}^m u_i(t) \widehat{\xi}_i(x, z).$$



Then  $x(t)$  is a trajectory in  $\mathbb{R}^n$  with the same length:

$$\text{length}(x(\cdot)) = \text{length}((x, z)(\cdot)) = \int \sqrt{\sum u_i^2(t)} dt.$$

It implies that  $\widehat{d}$  can be obtained from the sub-Riemannian distance  $d_{G_p}$  in  $G_p$  by

$$\widehat{d}(q_1, q_2) = \inf_{\tilde{q}_2 \in q_2 H_p} d_{G_p}(\tilde{q}_1, \tilde{q}_2), \quad \text{for any } \tilde{q}_1 \in q_1 H_p,$$

or, equivalently,  $B^{\widehat{d}}(q_1, \varepsilon) = \phi_p(B^{d_{G_p}}(\tilde{q}_1, \varepsilon))$ .

We will use this idea to desingularize the original space  $(M, d)$ . Choose for  $x$  privileged coordinates at  $p$ , so that

$$X_i(x) = \widehat{X}_i(x) + R_i(x) \quad \text{with } \text{ord}_p R_i \geq 0.$$

Set  $\widetilde{M} = M \times \mathbb{R}^{N-n}$ , and in local coordinates  $(x, z)$  on  $\widetilde{M}$ , define vector fields on a neighbourhood of  $(p, 0)$  as

$$\xi(x, z) = X_i(x) + \sum_{j=n+1}^N b_{ij}(x, z) \partial_{z_j},$$

with the same functions  $b_{ij}$  than in (15).

We define in this way a nonholonomic system on an open set  $\widetilde{U} \subset \widetilde{M}$  which nilpotent approximation at  $(p, 0)$  is, by construction, given by  $(\widehat{\xi}_1, \dots, \widehat{\xi}_m)$ . Unfortunately,  $(p, 0)$  can be itself a singular point. Indeed, a point can be singular for a system and regular for the nilpotent approximation taken at this point.

*Example 4.1.* Take the vector fields  $X_1 = \partial_{x_1}$ ,  $X_2 = \partial_{x_2} + x_1 \partial_{x_3} + x_1^2 \partial_{x_4}$  and  $X_3 = \partial_{x_5} + x_1^{100} \partial_{x_4}$  on  $\mathbb{R}^5$ . The origin 0 is a singular point. However the nilpotent approximation at 0 is  $\widehat{X}_1 = X_1$ ,  $\widehat{X}_2 = X_2$ ,  $\widehat{X}_3 = \partial_{x_5}$ , for which 0 is not singular.

To avoid this difficulty, we take a group bigger than  $G_p$ , namely the free nilpotent group  $N_r$  of step  $r$  with  $m$  generators. It is a Carnot group and its Lie algebra  $\mathfrak{n}_r$  is the free nilpotent Lie algebra of step  $r$  with  $m$  generators. The given of  $m$  generators  $\alpha_1, \dots, \alpha_m$  of  $\mathfrak{n}_r$  define on  $N_r$  a right-invariant sub-Riemannian distance  $d_N$ .

The group  $N_r$  can be thought as a group of diffeomorphisms and so define a left action on  $\mathbb{R}^n$ . Denoting by  $J$  the isotropy subgroup of 0 for this action, we obtain that  $(\mathbb{R}^n, \widehat{d})$  is isometric to  $N_r/J$  endowed with the restriction of the distance  $d_N$ .

By the same reasoning as above, we are able to lift locally the vector fields  $(X_1, \dots, X_m)$   $(\Delta, g)$  on  $M$  to vector fields on  $M \times \mathbb{R}^{\tilde{n}-n}$ ,  $\tilde{n} = \dim N_r$ , having  $\alpha_1, \dots, \alpha_m$  for nilpotent approximation at  $(p, 0)$ . And, because  $N_r$  is free up to step  $r$ , this implies that  $(p, 0)$  is a regular point for the associated nonholonomic system in  $M \times \mathbb{R}^{\tilde{n}-n}$ . We obtain in this way a result of desingularization.

**Lemma 4.5.** *Let  $p$  be a point in  $M$ ,  $r$  the degree of nonholonomy at  $p$ ,  $\tilde{n} = \tilde{n}(m, r)$  the dimension of the free Lie algebra of step  $r$  with  $m$  generators, and  $\tilde{M}$  the manifold  $\tilde{M} = M \times \mathbb{R}^{\tilde{n}-n}$ .*

*then there exist a neighbourhood  $\tilde{U} \subset \tilde{M}$  of  $(p, 0)$ ; a neighbourhood  $U \subset M$  of  $p$  with  $U \times \{0\} \subset \tilde{U}$ ; local coordinates  $(x, z)$  on  $\tilde{U}$ ; and smooth vector fields on  $\tilde{U}$ :*

$$\xi(x, z) = X_i(x) + \sum_{j=n+1}^N b_{ij}(x, z) \partial_{z_j},$$

*such that:*

- $\xi_1, \dots, \xi_m$  satisfy Chow's Condition and has  $r$  for degree of nonholonomy everywhere (so its Lie algebra is free up to step  $r$ );
- every  $\tilde{q}$  in  $\tilde{U}$  is regular;
- denoting  $\pi : \tilde{M} \rightarrow M$  the canonical projection and  $\tilde{d}$  the sub-Riemannian distance defined by  $\xi_1, \dots, \xi_m$  on  $\tilde{U}$ , we have, for  $q \in U$  and  $\varepsilon > 0$  small enough,

$$B(q, \varepsilon) = \pi \left( B^{\tilde{d}}((q, 0), \varepsilon) \right),$$

*or, equivalently,*

$$d(q_1, q_2) = \inf_{\tilde{q}_2 \in \pi^{-1}(q_2)} \tilde{d}((q_1, 0), \tilde{q}_2).$$

*Remark.* The lemma still holds if we take for  $r$  any integer greater than the degree of nonholonomy at  $p$ .

Thus any Carnot-Carathéodory space is locally the projection of an equiregular Carnot-Carathéodory space. This projection preserves the trajectories, the minimizers, and the distance.

### Application: uniform Ball-Box theorem

The key feature of regular points is the uniformity:

- uniformity of the flag (7);
- uniformity w.r.t.  $p$  of the convergence  $(\lambda(M, d), p) \rightarrow C_p M$  (as explained by Bellaïche [Bel96, Sect. 8], this uniformity is responsible for the group structure of the metric tangent space);
- uniformity of the distance estimates (see Remark page 24).

This last property in particular is essential to compute Hausdorff dimension or to prove the global convergence of approximate motion planning algorithms. Recall what we mean by uniformity in this context: in a neighbourhood of a regular point  $p_0$ , we can construct privileged coordinates depending continuously on the base point  $p$  and such that the distance estimate (11) holds with  $C_p$  and  $\varepsilon_p$  independent of  $p$ .

As already noticed, all these uniformity properties are lost at singular points. However, using the desingularization of the sub-Riemannian manifold, it is possible to give a uniform version of the distance estimates.

Let  $\Omega \subset M$  be a compact set. We denote by  $r_{\max}$  the maximum of the degree of nonholonomy on  $\Omega$ . We assume that  $M$  is an oriented manifold, so the determinant  $n$ -form  $\det$  is well-defined.

Given  $q \in \Omega$  and  $\varepsilon > 0$ , we consider the families  $\mathcal{X} = (X_{I_1}, \dots, X_{I_n})$  of brackets of length  $|I_i| \leq r_{\max}$ . On the finite set of these families, we define a function

$$f_{q,\varepsilon}(\mathcal{X}) = |\det (X_{I_1}(q)\varepsilon^{|I_1|}, \dots, X_{I_n}(q)\varepsilon^{|I_n|})|.$$

We say that  $\mathcal{X}$  is an *adapted frame* at  $(q, \varepsilon)$  if it achieves the maximum of  $f_{q,\varepsilon}$ .

The values at  $q$  of an adapted frame at  $(q, \varepsilon)$  clearly form a basis of  $T_q M$ . Moreover,  $q$  being fixed, the adapted frames at  $(q, \varepsilon)$  are adapted frames at  $q$  for  $\varepsilon$  small enough.

**Theorem 4.6** (Uniform Ball-Box theorem). *There exist positive constants  $K$  and  $\varepsilon_0$  such that, for all  $q \in \Omega$  and  $\varepsilon < \varepsilon_0$ , if  $\mathcal{X}$  is an adapted frame at  $(q, \varepsilon)$ , then*

$$\text{Box}_{\mathcal{X}}(q, \frac{1}{K}\varepsilon) \subset B(q, \varepsilon) \subset \text{Box}_{\mathcal{X}}(q, K\varepsilon),$$

where  $\text{Box}_{\mathcal{X}}(q, \varepsilon) = \{e^{x_1 X_{I_1}} \circ \dots \circ e^{x_n X_{I_n}}(q) : |x_i| \leq \varepsilon^{|I_i|}, 1 \leq i \leq n\}$ .

Of course,  $q$  being fixed, this estimate is equivalent to the one of Ball-Box theorem for  $\varepsilon$  smaller than some  $\varepsilon_1(q) > 0$ . However  $\varepsilon_1(q)$  can be infinitely close to 0 though the estimate here holds for  $\varepsilon$  smaller than  $\varepsilon_0$ , which is independent of  $q$ .

## References

- [ABB12] A. Agrachev, D. Barilari, and U. Boscain. Introduction to riemannian and sub-riemannian geometry (from hamiltonian viewpoint). In preparation, 2012.
- [AS87] A. A. Agrachev and A. V. Sarychev. Filtrations of a Lie algebra of vector fields and nilpotent approximations of control systems. *Dokl. Akad. Nauk SSSR*, 285:777–781, 1987. English transl.: *Soviet Math. Dokl.*, 36:104–108, 1988.
- [Bel96] A. Bellaïche. The tangent space in sub-Riemannian geometry. In A. Bellaïche and J.-J. Risler, editors, *Sub-Riemannian Geometry*, Progress in Mathematics. Birkhäuser, 1996.
- [DLPR12] Lance Drager, Jeffrey Lee, Efton Park, and Ken Richardson. Smooth distributions are finitely generated. *Annals of Global Analysis and Geometry*, 41:357–369, 2012. 10.1007/s10455-011-9287-8.
- [FJ03] E. Falbel and F. Jean. Measures of transverse paths in sub-Riemannian geometry. *Journal d’Analyse Mathématique*, 91:231–246, 2003.
- [Gab95] A. Gabrielov. Multiplicities of zeroes of polynomials on trajectories of polynomial vector fields and bounds on degree of non-holonomy. *Mathematical Research Letters*, 2:437–451, 1995.
- [Ger84] V. Ya. Gershkovich. Two-sided estimates of metrics generated by absolutely non-holonomic distributions on Riemannian manifolds. *Sov. Math. Dokl.*, 30:506–510, 1984.
- [GJR98] A. Gabrielov, F. Jean, and J.-J. Risler. Multiplicity of polynomials on trajectories of polynomials vector fields in  $\mathbb{C}^3$ . In W. Pawłucki B. Jakubczyk and J. Stasica, editors, *Singularities Symposium – Łojasiewicz 70*, volume 44, pages 109–121. Banach Center Publications, Warszawa, 1998.
- [Gro96] M. Gromov. Carnot-Carathéodory spaces seen from within. In A. Bellaïche and J.-J. Risler, editors, *Sub-Riemannian Geometry*, Progress in Mathematics. Birkhäuser, 1996.
- [Her91] H. Hermes. Nilpotent and high-order approximations of vector field systems. *SIAM Review*, 33(2):238–264, 1991.

- [Kup96] I. Kupka. Géométrie sous-riemannienne. In *Séminaire N. Bourbaki*, volume 817, June 1996.
- [LS91] G. Lafferriere and H. Sussmann. Motion planning for controllable systems without drift. In *Proceedings of the 1991 IEEE International Conference on Robotics and Automation*, Sacramento, California, 1991.
- [Mit85] J. Mitchell. On Carnot-Carathéodory metrics. *Journal of Differential Geom.*, 21:35–45, 1985.
- [MM00] G. A. Margulis and G. D. Mostow. Some remarks on the definition of tangent cones in a Carnot-Carathéodory space. *Journal d'Analyse Mathématique*, 80:299–317, 2000.
- [Mon02] R. Montgomery. *A tour of sub-Riemannian geometries, their geodesics and applications*. Math. Surveys and Monographs. American Math. Soc., 2002.
- [NSW85] A. Nagel, E. M. Stein, and S. Wainger. Metrics defined by vector fields. *Acta Math.*, 155:103–147, 1985.
- [RS76] L. P. Rothschild and E. M. Stein. Hypoelliptic differential operators and nilpotent groups. *Acta Math.*, 137:247–320, 1976.
- [Ste86] G. Stefani. On local controllability of a scalar-input system. In Lindquist Byrnes, editor, *Theory and Appl. of Nonlinear Control Syst.*, pages 167–179. North Holland, Amsterdam, 1986.